LIE ALGEBRAS: LECTURE 6
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The goal of representation theory is to study abstract Lie algebras by mapping them to matrix Lie algebras, where we can apply Linear Algebra techniques to study the image and gain information about the abstract Lie algebra.

1. Weyl's Theorem

Lemma 1.1. *Let $\phi : L \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra L. Then $\phi(L) \subset \mathfrak{sl}(V)$. In particular, L acts trivially on any one dimensional L-module.*

Proof. Since $L = [L, L]$ and $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$, we have

$$\phi(L) = \phi([L, L]) = [\phi(L), \phi(L)] \subset [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V).$$

The second claim follows from the fact that a 1×1-matrix with trace zero is identically zero. □

Let $\mathrm{Hom}(V, W)$ be the vector space of linear maps from V to W. If V and W are L-modules, then L acts naturally on $\mathrm{Hom}(V, W)$ as follows: for $x \in L$ and $f \in \mathrm{Hom}(V, W)$, let $(x.f)(v) := x.f(v) - f(x.v)$

Theorem 1.2 (Weyl's Theorem)**.** *Let $\phi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of a semisimple Lie algebra. Then ϕ is completely reducible.*

Proof. First we prove that an L-submodule W of codimension one has a complementary submodule in V. Now L acts trivially on the one dimensional module V/W, by the lemma. We denote this module by F. Then the sequence $0 \to W \to V \to F \to 0$ is an exact sequence of L-modules.

By induction on the dimension of W we will reduce to the case that W is irreducible. Suppose that W' is a proper non-zero submodule of W, then $0 \to W/W' \to V/W' \to F \to 0$ is an exact sequence of L-modules. By induction, this sequence "splits", that is, there exists a one dimensional submodule of V/W' which is complementary to W/W', call it $\tilde{W}/W'$. Then

$V/W' = W/W' \oplus \tilde{W}/W'$, and $0 \to W' \to \tilde{W} \to F \to 0$ is exact. By induction there is a one dimensional submodule X complementary to W' in $\tilde{W}$, so that $\tilde{W} = W' \oplus X$. Then $V = W \oplus X$ since $W \cap X = 0$ and $\dim W + \dim X = \dim V$.

Now we may assume that W is irreducible. (We may assume without loss of generality that L acts faithfully on V via ϕ, by replacing L with L', where $L = L' \oplus \ker\, \phi$.) Let c_ϕ be the Casimir element of ϕ. Since $c = c_\phi$ commutes with $\phi(L)$, c is an L-module endomorphism of V. So $c(W) \subset W$, and $\ker c$ is an L-submodule of V. Since L acts trivially on V/W, c must also act trivially since it is a linear combination of products of $\phi(x)$ with $x \in L$. So c has trace zero on V/W. By Schur's Lemma, c acts by a scalar on the irreducible submodule W, which cannot also be zero since $\mathrm{Tr}_V(c) = \dim L \neq 0$. Hence, the kernel of c is a one dimensional L-submodule which intersects W trivially. Hence, $V = W \oplus \ker\, c$.

Now we handle the general case. Let W be a proper non-zero submodule of V. Then $0 \to W \to V \to V/W \to 0$ is exact. Let $\mathrm{Hom}(V, W)$ be the space of linear maps from V to W, viewed as an L-module. Let $\mathcal{V}$ be the subspace of $\mathrm{Hom}(V, W)$ consisting of the maps whose restriction to W is just scalar multiplication. Then $\mathcal{V}$ is a submodule. Suppose $f\mid_W = a \cdot 1_W$, then for $x \in L$, $w \in W$, $(x.f)(w) = x.f(w) - f(x.w) = a(x.w) - a(x.w) = 0$. Thus $(x.f)\mid_W = 0$. Let $\mathcal{W}$ be the subspace of $\mathcal{V}$ consisting of the maps whose restriction of W is zero. By the previous calculation, $\mathcal{W}$ is also a submodule. Also, $\mathcal{V}/\mathcal{W}$ has dimension one, since each $f \in \mathcal{V}$ is determined modulo $\mathcal{W}$ by the scalar $f\mid_W$. So we have an exact sequence of L-modules $0 \to \mathcal{W} \to \mathcal{V} \to F \to 0$.

By the first part of the proof, $\mathcal{W}$ has a one dimensional complement submodule in $\mathcal{V}$. Let $f : V \to W$ span this submodule, and after multiplying by a scalar we may assume that $f\mid_W = 1_W$. Since this spans a one dimensional submodule, L acts trivially. So for all $x \in L$ and $v \in V$, we have $0 = (x.f)(v) = x.f(v) - f(x.v)$. Hence, f is an L-module homomorphism from V to W. Hence, $\ker f$ is an L-submodule of V. Since f maps V into W and acts by 1_W on W, we conclude that $V = W \oplus \ker\, f$. $\square$

2. $\mathfrak{sl}_2(\mathbb{F})$ IS SIMPLE

Let V be a finite dimensional vector space over a field $\mathbb{F}$, and $f \in \text{End } V$. We call f *semisimple* if the roots of its minimal polynomial are all distinct. When $\mathbb{F}$ is algebraically closed, this is equivalent to the condition that f is *diagonalizable* (i.e. there exists a basis for V in which f is a diagonal matrix). If f is diagonalizable, then $V = \oplus_{\lambda \in S} V_\lambda$, where $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$ is the eigenspace of the eigenvalue λ, and S is the set of eigenvalues of f.

Lemma 2.1. *Let $f \in End(V)$ be diagonalizable and let W be an f-invariant subspace of V (i.e. $f(W) \subset W$). Then*

$$W = \oplus_{\lambda \in S} W_\lambda \text{ where } W_\lambda = W \cap V_\lambda.$$

Proof. Clearly $\oplus_{\lambda \in S} W_\lambda \subset W$. Now let $x \in W$. Then $x = \sum_{\lambda \in S} x_\lambda$ with $x_\lambda \in V_\lambda$. We need to show that $x_\lambda \in W$ for each $\lambda \in S$. Now $f^k(x) \in W$ for each k, so

$$\sum_{\lambda \in S} \lambda^k x_\lambda = \sum_{\lambda \in S} f^k(x_\lambda) = f^k(x) \in W.$$

Let $T = \{\lambda \in S \mid x_\lambda \neq 0\}$ and $t = |T|$, so that $T = \{\lambda_1, \ldots, \lambda_t\}$. Then

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_t \\ \vdots & \vdots & \ldots & \vdots \\ \lambda_1^{t-1} & \lambda_2^{t-1} & \ldots & \lambda_t^{t-1} \end{pmatrix} \begin{pmatrix} x_{\lambda_1} \\ x_{\lambda_2} \\ \vdots \\ x_{\lambda_t} \end{pmatrix} = \begin{pmatrix} x \\ f(x) \\ \vdots \\ f^{t-1}(x) \end{pmatrix}.$$

The determinant of the Vandermonde matrix is

$$\Pi_{i<j}(\lambda_i - \lambda_j) \neq 0,$$

and hence it is invertible. Therefore, each x_λ can be expressed as a linear combination of the elements $x, f(x), \ldots, f^{k-1}(x) \in W$. $\square$

Lemma 2.2. *The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ is simple.*

Proof. We now have two ways to see that $\mathfrak{sl}_2(\mathbb{F})$ is simple. The first method is to use the 4th homework problem from the third assignment, which states that a 3-dimensional Lie algebra L with $L = [L, L]$ is simple. The second method is to prove it directly using the above lemma. Let $\{e, h, f\}$ be the standard basis for $\mathfrak{sl}_2(\mathbb{F})$. This is an eigenbasis of L for $\text{ad}_h \in \mathfrak{gl}(L)$. In particular, ad_h is diagonalizable. An ideal I of L (a submodule of the adjoint action) should thus be spanned by a subset of $\{e, h, f\}$. It is easy to check that it not possible to have a non-empty proper subset span an ideal. $\square$

3. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

By Weyl's Theorem, all finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ are completely reducible. So we only need to study the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ to describe all finite dimensional representations.

Let e, h, f be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. Then $[h, x] = 2x$, $[h, y] = -2y$, and $[xy] = h$. A (non-zero) vector v of an $\mathfrak{sl}_2$-module V is called a *weight vector* if $h.v \in \mathbb{C}v$. The vector v is said to be *of weight* λ ($\lambda \in \mathbb{C}$) if $h.v = \lambda v$. The vector v is called *primitive* if $e.v = 0$.

Lemma 3.1. *If $v \in V_\lambda$, then $e.v \in V_{\lambda+2}$ and $f.v \in V_{\lambda-2}$.*

Proof. We have that $h.(e.v) = [h, e].v + e.(h.v) = 2e.v + \lambda e.v = (\lambda + 2)e.v$, and $h.(f.v) = [h, f].v + f.(h.v) = -2f.v + \lambda f.v = (\lambda - 2)f.v$. $\square$

Lemma 3.2. *A finite dimensional $\mathfrak{sl}_2$-module has a primitive vector.*

Proof. Let V be an finite dimensional $\mathfrak{sl}_2$-module. Then the endomorphism $h : V \to V$ has a (non-zero) eigenvector $v \in V_\lambda$ for some $\lambda \in \mathbb{C}$. By the above lemma, $e^k.v \in V_{\lambda+2k}$. Since V is finite dimensional and nonzero eigenvectors are linearly independent, we must have that $e^k.v = 0$ for some k. Choose $k \in \mathbb{N}$ such that $e^k.v \neq 0$ but $e^{k+1}.v = 0$. Then $e^k.v$ is a primitive vector for the module V. $\square$

Let V be an irreducible finite dimensional $\mathfrak{sl}_2$-module, and let v_0 be a primitive vector. Set $v_n = f^n.v_0$.

Lemma 3.3. *For $n \geq 0$,*
(a) $h.v_n = (\lambda - 2n)v_n$,
(b) $f.v_n = v_{n+1}$,
(c) $e.v_n = n(\lambda + 1 - n)v_{n-1}$.

Proof. Part (a) follows from simple induction using Lemma 3.1, and (b) is true by definition. Now $e.v_0 = 0$ since v_0 is a primitive vector. Now suppose that $e.v_{n-1} = (n - 1)(\lambda + 1 - (n - 1))v_{n-2}$. Then

$$\begin{aligned} e.v_n &= e.f.v_{n-1} \\ &= [e, f].v_{n-1} + f.e.v_{n-1} \\ &= h.v_{n-1} + (n - 1)(\lambda + 1 - (n - 1))f.v_{n-2} \\ &= (\lambda - 2(n - 1))v_{n-1} + (n - 1)(\lambda - (n - 2))v_{n-1} \\ &= n(\lambda + 1 - n)v_{n-1}. \end{aligned}$$

□

Claim 3.4. $V = span\{v_0, v_1, \ldots, v_m\}$

By part (a), the nonzero vectors v_n are linearly independent. Since V is finite dimensional, there is an integer m such that $v_m \neq 0$ and $v_{m+1} = 0$. Then the subspace of V with basis $\{v_0, v_1, \ldots, v_m\}$ is a non-zero L-submodule of V. Since V is irreducible, this subspace must be all of V.

Claim 3.5. $\lambda = m$ *for some* $m \in \mathbb{Z}$.

Now by part (c) of the previous lemma,

$$0 = e.v_{m+1} = (m+1)(\lambda + 1 - (m+1))v_m,$$

which implies that $\lambda = m \in \mathbb{N}$. Thus, $h.v_0 = mv_0$ where m is a non-negative integer. The weight of the maximal vector is called the *highest weight* of V. Thus, $V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$, where each weight space has dimension one. The module V is determined uniquely by m since $\dim V = m + 1$.

Claim 3.6. V *is irreducible*

The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is determined by the formulas given above. The fact that this module is irreducible follows from Lemma 2.1. Since h is diagonalizable on V, it is also diagonalizable on a submodule. So a submodule must be spanned by a subset of the v_n, but this is impossible for a proper subset. Hence, we have proven:

Theorem 3.7. *Let* V *be an irreducible finite dimensional representation of* $\mathfrak{sl}_2(\mathbb{C})$, *and let* v_0 *be a primitive vector of weight* λ. *Then* $\lambda \in \mathbb{N}$. *The set* $\{v_0, v_1, \ldots, v_m\}$ *is a vector space basis of* V, *so* $\dim V = \lambda + 1$. *The module structure of* V *is given by the above formulas.*

It remains to check whether there exists an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of each possible highest weight $m = 0, 1, 2, \ldots$. It suffices to show that the equations above define an $\mathfrak{sl}_2(\mathbb{C})$-module structure on the vector space $V(m)$ with basis $\{v_0, v_1, \ldots, v_m\}$. For this one should check that the matrices defined by these equations satisfy the structural equations for $\mathfrak{sl}_2(\mathbb{C})$.

Claim 3.8. *If V is a finite dimensional $\mathfrak{sl}_2$-module, then the eigenvalues of h are all integers and* $\dim V_\mu = \dim V_{-\mu}$. *Moreover,the number of irreducible summands is* $\dim V_0 + \dim V_1$. *For each $m \in \mathbb{N}$, the multiplicity of each $V(m)$ in V is equal to* $\dim V_m - \dim V_{m+2}$.

By Weyl's Theorem V is a direct sum of irreducible modules as described above. Suppose W is an irreducible module with highest weight m for some nonnegative integer m. Then eigenspaces of W are all integers and $W = W_{-m} \oplus W_{-m+2} \oplus \cdots \oplus W_{m-2} \oplus W_m$.

Notation: We have shown that the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $V(m)$ has a basis $\{v_0, v_1, \ldots, v_m\}$ with the action of $\mathfrak{sl}_2(\mathbb{C})$ defined by the equations in Lemma 3.3. Denote $V = V(m)$. We have also shown that $V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$, so it is convenient to relabel this basis so that $v_k \in V_k$ with $k \in \{-m, -m+2, \ldots, m-2, m\}$. Then the equations in Lemma 3.3 become

$$(a)\ h.v_k = kv_k$$
$$(b)\ f.v_k = v_{k-2}$$
$$(c)\ e.v_k = \left(\frac{m-k}{2}\right)\left(\frac{m+k+2}{2}\right) v_{k+2}.$$

Example 3.9. We will compute the irreducible submodules of the $\mathfrak{sl}_2(\mathbb{C})$-module $M := V(1) \otimes V(1)$. Since $\{v_1, v_{-1}\}$ is a basis for $V(1)$, we have that $\{v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1}\}$ is a basis for M. Now

$$h.(v_i \otimes v_j) = h.v_i \otimes v_j + v_i \otimes h.v_j = (i+j)(v_i \otimes v_j).$$

Thus $M = M_{-2} \oplus M_0 \oplus M_2$, with $M_{-2} = span\{v_{-1} \otimes v_{-1}\}$, $M_0 = span\{v_1 \otimes v_{-1},\ v_{-1} \otimes v_1\}$, and $M_2 = span\{v_1 \otimes v_1\}$. Hence,

$$V(1) \otimes V(1) = V(2) \oplus V(0).$$

One can check that in this decomposition $V(0) = span\{v_1 \otimes v_{-1} - v_{-1} \otimes v_1\}$ and $V(2) = span\{v_1 \otimes v_1, v_1 \otimes v_{-1} + v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$.

4. Construction of irreducible $\mathfrak{sl}_2$-modules

Finally, we construct a natural irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, $V(m)$, of dimension $m+1$, for each $m \in \mathbb{N}$. Now $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{gl}_2(\mathbb{C}) = \langle E_{11}, E_{12}, E_{21}, E_{22}\rangle$, where E_{ij} denotes the matrix whose only non-zero entry is 1 in the ij position. Then

$$e = E_{12}, \qquad h = E_{11} - E_{22}, \qquad f = E_{21}.$$

Let $\mathbb{C}[x, y]$ be the polynomial algebra in two variables, and define an action of $\mathfrak{gl}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$ as follows:

$$E_{11}(p) = xp'_x, \quad E_{22}(p) = yp'_y, \quad E_{12}(p) = xp'_y, \quad E_{21} = yp'_x,$$

where p'_x and p'_y is the partial derivative with respect to x and y, respectively.

Lemma 4.1. *These formulas define a $\mathfrak{gl}_2(\mathbb{C})$-module structure on $\mathbb{C}[x, y]$.*

Proof. This can be checked by direct calculation. Alternatively, note that a derivation of $\mathbb{C}[x, y]$ is determined by its value on x and y. If $d(x) = p$ and $d(y) = q$, then $d(f) = pf'_x + qf'_y$. So it suffices to check that the brackets on x and y. But when we restrict to $\langle x, y\rangle$, this is the natural representation of $\mathfrak{gl}_2(\mathbb{C})$, which is well-defined. □

The set of homogeneous polynomials of degree m is an $\mathfrak{gl}_2(\mathbb{C})$-submodule and an $\mathfrak{sl}_2(\mathbb{C})$-submodule. Denote this submodule by $W(m)$. Then we have that $W(m) = span\{x^m, x^{m-1}y, x^{m-2}y^2, \ldots, xy^{m-1}, y^m\}$. One can check that this module is simple by using Lemma 2.1.