LIE ALGEBRAS: LECTURE 8
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1. Root space decomposition (continued)

Let $\mathbb{F}$ be an algebraically closed field with characteristic zero. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$, and let $\mathfrak{h}$ be a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\ \mathfrak{g}_\alpha)$$

be the root space decomposition, where $\Delta \subset \mathfrak{h}^*$ is the set of roots of the $\mathfrak{g}$.

Recall that for $\alpha \in \mathfrak{h}^*$ we define $t_\alpha \in \mathfrak{h}$ to be the unique element such that $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in \mathfrak{h}$. We define $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$. Then if $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) = 1$, then $\{x_\alpha, h_\alpha, y_\alpha\}$ are a basis for a subalgebra S_α which is isomorphic to the standard $\mathfrak{sl}_2$. Last time, we showed that for each α there exists a subalgebra of this form. The next proposition implies that this subalgebra is uniquely determined for each $\alpha \in \Delta$.

Proposition 1.1. *If* $\alpha \in \Delta$, *then* $\dim \mathfrak{g}_\alpha = 1$ *and* $\mathbb{F}\alpha \cap \Delta = \{\pm\alpha\}$.

Proof. Fix $\alpha \in \Delta$, and choose $S_\alpha \cong \mathfrak{sl}(2)$ (see Proposition 2.6 in Lecture 7) Let

$$M = \mathfrak{h} \oplus \left(\oplus_{c\in\mathbb{F}\setminus\{0\}}\ \mathfrak{g}_{c\alpha}\right).$$

Then M is a finite dimensional S_α module. Hence, it decomposes into a direct sum of irreducible modules. The weights are 0 (with multiplicity $\dim \mathfrak{h}$) and $c\alpha(h_\alpha) = 2c$. Since these must be integers, we have that $c \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$.

Now $\operatorname{Ker}\alpha = \{h \in \mathfrak{h} \mid \alpha(h) = 0\}$ is a subspace of codimension 1 in $\mathfrak{h}$, and S_α acts trivially on $\operatorname{Ker}\alpha$ for $h \in \operatorname{Ker}\alpha$ since

$$[x_\alpha, h] = -\alpha(h)x_\alpha = 0.$$

Also, S_α is an irreducible S_α-submodule of M. Now $\operatorname{Ker}\alpha$ and S_α exhaust all occurrences of the weight 0 in the module M. Hence, the only even weights occuring in M are 0 and 2. In particular, $c \neq 2$ which implies that 2α is never a root. Equivalently, $\frac{\alpha}{2}$ is never a root. Hence, 1 is not a weight. Thus we have found all of the weights of M occurring in $\operatorname{Ker}\alpha$ and S_α. Therefore, $M = \mathfrak{h} + S_\alpha$ and the proposition follows. $\square$

Set $\langle\beta,\alpha\rangle := \beta(h_\alpha)$. Note that $\langle\cdot,\cdot\rangle$ is linear in the first argument.

Proposition 1.2. *If $\alpha,\beta \in \Delta$, then $\langle\beta,\alpha\rangle \in \mathbb{Z}$ and $\beta - <\beta,\alpha> \alpha \in \Delta$. If $\beta \neq \pm\alpha$, let r, q be the largest integers for which $\beta - r\alpha$, $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \Delta$ for $-r \leq i \leq q$, and $<\beta,\alpha>= r - q \in \mathbb{Z}$. (Note that this is called an α-*string through β.*) Thus, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. Fix $\alpha,\beta \in \Delta$, and choose $S_\alpha \cong \mathfrak{sl}(2)$. The proposition is clear if $\beta = \pm\alpha$, so suppose that $\beta \neq \pm\alpha$. Set $K = \sum_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$. Then K is a S_α-submodule of $\mathfrak{g}$ with one dimensional weight spaces and integral weights $\langle\beta,\alpha\rangle + 2i$. Thus, $\langle\beta,\alpha\rangle \in \mathbb{Z}$. Since 0 and 1 can not both occur as weights, K is irreducible. By $\mathfrak{sl}(2)$ theory, the weights are $\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha$. Since $(\beta - r\alpha)(h_\alpha) = -(\beta + q\alpha)(h_\alpha)$ we have that

$$\langle\beta,\alpha\rangle = \beta(h_\alpha) = r - q.$$

Since $\beta + i\alpha \in \Delta$ when $-r \leq i \leq q$, we conclude that

$$\beta - <\beta,\alpha> \alpha = \beta + (q - r)\alpha \in \Delta.$$

$\square$

2. Euclidean space

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Recall, that since the Killing form is non-degenerate when restricted to $\mathfrak{h}$, we may define a pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ as follows: for $\lambda \in \mathfrak{h}^*$ define t_λ to be the unique element of $\mathfrak{h}$ satisfying $\lambda(h) = \kappa(t_\lambda, h)$ for all $h \in \mathfrak{h}$. Then we have a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ defined by

$$(\lambda,\mu) := \kappa(t_\lambda, t_\mu).$$

Then define $\langle\beta,\alpha\rangle := \beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Note that $\langle\beta,\alpha\rangle$ is not symmetric. Since Δ spans $\mathfrak{h}^*$, we can choose a basis $\{\alpha_1, \ldots, \alpha_t\} \subset \Delta$ for $\mathfrak{h}^*$.

Lemma 2.1. *Let E_Q be the rational span of $\{\alpha_1, \ldots, \alpha_t\}$, then $\Delta \subset E_Q$ and $\dim_{\mathbb{Q}} E_Q = t$.*

Proof. If $\beta \in \Delta$, then we can write β uniquely as $\beta = \sum_{i=1}^{t} c_i\alpha_i$. We claim that $c_i \in \mathbb{Q}$. Now for $i = 1, \ldots, t$,

$$(\beta, \alpha_j) = \sum_{i=1}^{t} c_i(\alpha_i, \alpha_j)$$

By multiplying by $\frac{2}{(\alpha_j,\alpha_j)}$ we obtain for each $j = 1, \ldots, t$:

$$\langle \beta, \alpha_j \rangle = \sum_{i=1}^{t} \langle \alpha_i, \alpha_j \rangle c_i$$

Since $\{\alpha_1, \ldots, \alpha_t\}$ is a basis and the form is non-degenerate, the matrix $A_{ij} := (\langle \alpha_i, \alpha_j \rangle)$ is non-singular and hence invertible. Since this is a system of integral equations, the solution will be rational. Therefore, $c_i \in \mathbb{Q}$. □

Lemma 2.2. *The form $(-,-)$ is a positive definite symmetric bilinear form on E_Q, .*

Proof. The form $(-,-)$ is symmetric and bilinear by definition, because the Killing form is symmetric and bilinear. First, we must show that for any $\lambda, \mu \in E_Q$ we have $(\lambda, \mu) \in \mathbb{Q}$. It suffices to show this is true for any $\alpha, \beta \in \Delta$ since E_Q is defined to be the rational span of the roots $\{\alpha_1, \ldots, \alpha_t\}$ and the form $(-,-)$ is bilinear. Now we have that for $\lambda, \mu \in E_Q$,

$$(\lambda, \mu) = Tr(\text{ad } t_\lambda \text{ad } t_\mu) = \sum_{\alpha \in \Delta} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha \in \Delta} (\alpha, \lambda)(\alpha, \mu).$$

In particular, for $\beta \in \Delta$

$$(\beta, \beta) = \sum_{\alpha \in \Delta} (\alpha, \beta)^2.$$

Then by multiply both sides by $\frac{4}{(\beta,\beta)^2}$ we obtain

$$\frac{4}{(\beta, \beta)} = \sum_{\alpha \in \Delta} \langle \alpha, \beta \rangle^2 \in \mathbb{Z}.$$

Hence, $(\beta, \beta) \in \mathbb{Q}$ for all $\beta \in \Delta$. Then for $\alpha, \beta \in \Delta$,

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = \langle \alpha, \beta \rangle \in \mathbb{Z}$$

implies that $(\alpha, \beta) \in \mathbb{Q}$.

So for $\lambda \in E_Q$, we have $(\lambda, \lambda) = \sum_{\alpha \in \Delta} (\alpha, \lambda)^2 \geq 0$. If $\lambda \neq 0$, this must be strict inequality since the Killing form is non-degenerate on $\mathfrak{h}$, and Δ spans $\mathfrak{h}$. Therefore, $(-, -)$ is positive definite. $\square$

Let E be the real vector space obtained by extending the base field of E_Q to $\mathbb{R}$ ($E := \mathbb{R} \otimes_{\mathbb{Q}} E_Q$). Then E is an inner product space, i.e. a finite dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form.

Lemma 2.3. *A real finite dimensional inner product space E is a Euclidean space (i.e. a real vector space with the dot product).*

Proof. Let E be a real finite dimensional inner product space E with inner product $(-, -)$. Then since $(-, -)$ is positive definite we can define length on E by $||v|| = \sqrt{(v, v)}$. We define the distance between two vectors v, w to be $||v - w||$. The Cauchy-Schwartz inequality follows: $|(v, w)| \leq ||v|| \, ||w||$. (For $y \neq 0$ by let $r = (y, y)^{-1}(x, y)$ and use the fact that $0 \leq (x - ry, x - ry)$.) We define the angle between two vectors x, y to be

$$\theta = \cos^{-1}\left(\frac{(v, w)}{||v|| \, ||w||}\right)$$

with $0 \leq \theta < \Pi$, which is well defined by the Cauchy-Schwartz inequality. Since E is finite dimensional, we can use the Gram-Schmidt process to find an orthonormal basis $\{v_1, \ldots, v_n\}$ with respect to the the inner product. Then $(v_i, v_j) = \delta_{ij}$, and in this basis $(-, -)$ is the dot product and E is a Euclidean space. $\square$

In summary, we have proven:

Theorem 2.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let Δ be the set of roots for $\mathfrak{h}$. Then there exists a Euclidean space E with $E \subset \mathfrak{h}^*$ and $\dim E = \dim \mathfrak{h}$. In addition,*

(1) *Δ is finite, spans E, and $0 \notin \Delta$.*
(2) *If $\alpha \in \Delta$, then $\mathbb{Z}\alpha \cap \Delta = \{\pm\alpha\}$.*
(3) *For all $\alpha, \beta \in \Delta$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$.*
(4) *If $\alpha, \beta \in \Delta$, then $\beta - \langle \beta, \alpha \rangle \alpha \in \Delta$.*

3. Abstract root systems

Let E be a Euclidean space with inner product $(\cdot,\cdot)$. Define $\langle\beta,\alpha\rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ for $\alpha,\beta \in E$. A non-zero vector $\alpha \in E$ determines a reflection $\sigma_\alpha \in \mathrm{GL}(E)$ defined by

$$\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha.$$

Then σ_α fixes the hyperplane $P_\alpha = \{\beta \in E \mid (\beta,\alpha) = 0\}$ and sends α to $-\alpha$. A subset Δ in a Euclidean space E is called a *root system* if it satisfies the following axioms:

(1) Δ is finite, spans E, and does not contain 0.
(2) If $\alpha \in \Delta$, then $\Delta \cap \mathbb{Z}\alpha = \{\pm\alpha\}$.
(3) For all $\alpha,\beta \in \Delta$, $\langle\beta,\alpha\rangle \in \mathbb{Z}$.
(4) If $\alpha \in \Delta$, then σ_α leaves Δ invariant.

Elements of Δ are called *roots*. The dimension of E is called the *rank* of Δ.

The *Weyl group*, denoted by W, is the subgroup of $GL(E)$ generated by the reflections σ_α for $\alpha \in \Delta$.

Lemma 3.1. *Let Δ be a root system in E, with Weyl group W. If $\sigma \in GL(E)$ leaves Δ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Delta$, and $\langle\beta,\alpha\rangle = \langle\sigma(\beta),\sigma(\alpha)\rangle$.*

Proof. First we show that if σ is refection preserving the root system Δ and sending α to $-\alpha$ for some $\alpha \in \Delta$, then $\sigma = \sigma_\alpha$. Let $\tau = \sigma\sigma_\alpha$. Then τ preserves Δ and $\tau(\alpha) = \alpha$. Now τ acts as the identity on $\mathbb{R}\alpha$ and on $E/\mathbb{R}\alpha$, so all eigenvalues of τ are equal to 1. Because Δ is finite, there is some integer $n \geq 1$ such that $\tau(\beta) = \beta$ for all $\beta \in \Delta$. Since Δ spans E, this implies $\tau^n = 1$. Hence, τ is diagonalizable with diagonal entries equal to 1. Therefore, $\tau = 1$ implying $\sigma = \sigma_\alpha$.

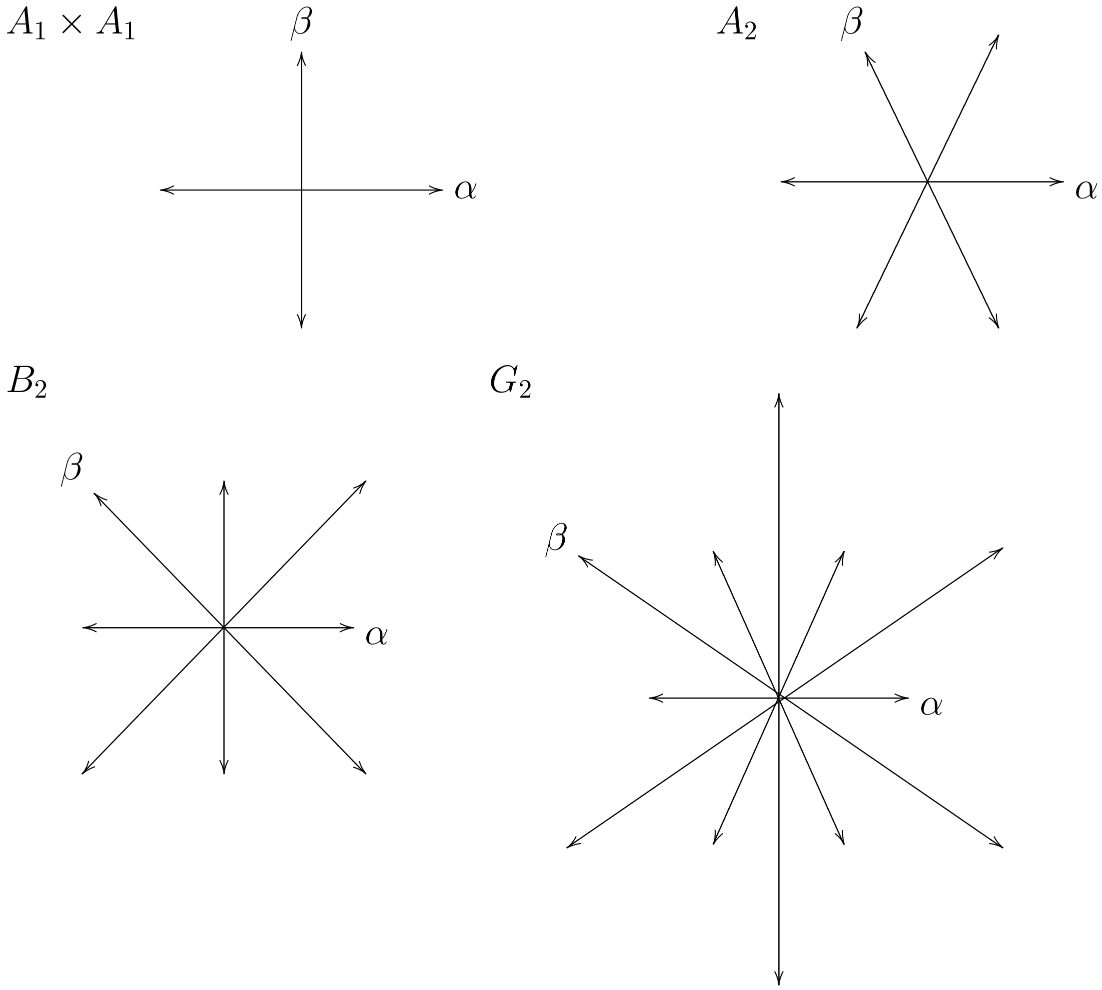
Next observe that $\sigma\sigma_\alpha\sigma^{-1}$ is a reflection preserving Δ which sends $\sigma(\alpha)$ to $-\sigma(\alpha)$, and hence $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$. Now $\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle\sigma(\beta),\sigma(\alpha)\rangle\sigma(\alpha)$ and $\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\sigma_\alpha(\beta)) = \sigma(\beta - \langle\beta,\alpha\rangle\alpha) = \sigma(\beta) - \langle\beta,\alpha,\rangle\sigma(\alpha)$. Hence, $\langle\beta,\alpha\rangle = \langle\sigma(\beta),\sigma(\alpha)\rangle$. ☐

There is only on root system of rank 1, namely

$$A_1 \quad \longleftrightarrow \quad .$$

This is the root system of $\mathfrak{sl}_2$.

Lemma 3.2. *The root systems of rank 2 are $A_1 \times A_1$, A_2, B_2, and G_2, as depicted below.*



Proof. First we determine the admissible angles between any two roots α, β. Now $(\alpha, \beta) = ||\alpha||\ ||\beta||\cos\,\theta$ implies that

$$\langle \alpha, \beta \rangle\ \langle \beta, \alpha \rangle = 4\cos^2\theta.$$

This is a non-negative integer between 0 and 3, because $< \alpha, \beta > \in \mathbb{Z}$ and $0 \leq \cos^2\theta < 1$. This equation determines the possible values for $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$ and θ. The following equation determines the ratio of lengths of these roots:

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{||\beta||}{||\alpha||}\cos\theta.$$

The possibilities with $\alpha \neq \pm\beta$, $||\beta|| \geq ||\alpha||$ and $0 \leq \theta < \Pi$ are listed in the following table.

Table 1

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\frac{\|\|\beta\|\|^2}{\|\|\alpha\|\|^2}$
0	0	$\frac{\Pi}{2}$	undefined
1	1	$\frac{\Pi}{3}$	1
-1	-1	$\frac{2\Pi}{3}$	1
1	2	$\frac{\Pi}{4}$	2
-1	-2	$\frac{3\Pi}{4}$	2
1	3	$\frac{\Pi}{6}$	3
-1	-3	$\frac{5\Pi}{6}$	3

Finally, we use the fact that the diagram is invariant under the reflections of roots (the Weyl group) to find the remaining roots of a diagram. (In particular, the reflections preserve root length.) □

Application: We have shown that if a semisimple Lie algebra has a Cartan subalgebra with dimension 2, then it has one of the roots systems listed above. We have not shown that there exists a Lie algebra for each of these abstract root systems.

The Weyl groups of $A_1 \times A_1$, A_2, B_2, G_2 (respectively) are dihedral groups of order 4, 6, 8, 12.

Lemma 3.3. *Let α and β be two non-proportional roots. If $\langle \alpha, \beta \rangle > 0$ (i.e. if $(\alpha, \beta) > 0$, the angle between α and β is strictly acute), then $\alpha - \beta$ is a root. If $\langle \alpha, \beta \rangle < 0$ (i.e. if $(\alpha, \beta) < 0$, the angle between α and β is strictly obtuse), then $\alpha + \beta$ is a root.*

Proof. The second statement follows from the first, by replacing β with $-\beta$. The proof of this lemma follows from the classification proof for rank 2 root systems. We see that when $\langle \alpha, \beta \rangle > 0$, either $\langle \alpha, \beta \rangle = 1$ or $\langle \beta, \alpha \rangle = 1$ (See page 45 of Humphreys). Since σ_α and σ_β leave Δ invariant, we have that $\beta - \langle \beta, \alpha \rangle \alpha \in \Delta$ and $\alpha - \langle \alpha, \beta \rangle \beta \in \Delta$. Hence, $\alpha - \beta \in \Delta$. $\square$