Lecture 1: Projection Games and Unique Games

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1 Projection Games and Unique Games

A *projection game* is as follows. Given is a bipartite graph $G = (A, B, E)$ where A and B are the two sets of vertices and Σ_A, Σ_B are two finite sets of lables. For each vertex $a \in A$ we have a function v_a, $v_a : \Sigma_A \to \{\text{TRUE}, \text{FALSE}\}$, indicating whether a label to a is legal. For each edge $e = (a, b) \in E$, we have a function π_e, $\pi_e : \Sigma_A \to \Sigma_B$ which gives a projection from a label to a to a label to b.

We have two players A and B that do not communicate. A verifier picks an edge (a, b) at random and asks player A for a label for a and player B for a label to b. We say that e is *satisfied* by labels $\sigma_a \in \Sigma_A, \sigma_b \in \Sigma_B$ if $v_a(\sigma_a) = \text{TRUE}$ and $\pi_e(\sigma_a) = \sigma_b$.

The goal of the players in the game is to find labelings $f_A : A \to \Sigma_A, f_B : B \to \Sigma_B$ to maximize

$$\mathop{\mathbf{Pr}}_{e=(a,b)\in E}[e \text{ is satisfied by } f_A(a), f_B(b)]$$

This maximum is called the *value* of the game.

A *unique game* is a projection game where $\Sigma_A = \Sigma_B := \Sigma$, $v_a \equiv \text{TRUE}\ \forall a \in A$ and all the projections are permutations. (i.e. one to one and onto function)

DEFINITION 1 (LABEL COVER/UNIQUE LABEL COVER) *The algorithmic problem of computing the value of a given projection game is called* Label Cover. *Similarly, for unique games, we get* Unique Label Cover.

2 The NP-hardness of Label Cover and Unique Label Cover

We will show the following theorem:

THEOREM 2
Label Cover and Unique Label Cover are **NP***-hard.*

We will in fact show the stronger claim that any constraint satisfaction problem (CSP) can be reduced to label cover (and some reduce to unique label cover).

DEFINITION 3 (D-CSP) *Given is a set of variables V, a finite alphabet Σ and a set $\mathcal{C}$ of constraints, each depending on D of the variables. The goal is to find an assignment to the variables to maximize the fraction of constraints satisfied.*

Here are some examples:

3SAT Given V = a set of boolean variables, $\Sigma = \{\text{TRUE}, \text{FALSE}\}$, $\mathcal{C}$ is a set of clauses, each depending on 3 variables.

Max-Cut Here V is a set of vertices in a graph, $\Sigma = \{0, 1\}$, $\mathcal{C}$ consists of a constraint for every edge that is satisfied iff the two labels are unequal.

Coloring Here, V is a set of vertices in a graph, Σ is a set of colors and $\mathcal{C}$ is a set of constraints, one for each edge that is satisfied iff the edge is not monochromatic.

All the above examples are **NP**-hard. We will show that any constraint satisfaction problem can be reduced to Label-Cover. We will have $A = \mathcal{C}$ and $B = V$. We add an edge from a constraint C to a variable v iff v appears in C. Also, $\Sigma_A = \Sigma^k, \Sigma_B = \Sigma$. We define $v_C(\sigma_1, \ldots, \sigma_k)$ to be true iff $\sigma_1, \ldots, \sigma_k$ is a satisfying assignment for the constraint.

We define the projection for each edge $e = (C, v) \in E$ to be $\pi_e(\sigma_1, \ldots, \sigma_k) = \sigma_i$ if v is the i^{th} variable that appears in C (we assign an ordering to the variables arbitrarily).

We can adopt the above reduction to give us *Unique Label-Cover* for *MAX-CUT*.

1. $\Sigma_A = \Sigma$
2. We let $\pi_e(\sigma) = \sigma$ if v is the first endpoint of e, and $1 - \sigma$ otherwise.

3 Approximating Label Cover and Unique Label Cover

DEFINITION 4 (α-APPROXIMATION) *An algorithm A gives α-approximation to a maximization problem P if for any input x,*

$$\alpha \cdot OPT(x) \leq A(x) \leq OPT(x)$$

LEMMA 5
There is an efficient algorithm giving $1/|\Sigma_B|$ approximation for Label-Cover.

PROOF: We can assume WLOG that for every vertex $a \in A$, there is some label $\sigma_a \in \Sigma_A$ such that $v_a(\sigma_a) = \text{TRUE}$.

For every vertex $b \in B$, pick a label $\sigma_b \in \Sigma_B$ that satisfies as many of the edges touching B as possible. This label must satisfy at least $1/|\Sigma_B|$ fraction of the edges touching b. Thus, counting over all the vertices b, we satisfy at least $1/|\Sigma_B|$ fraction of all edges. □

Note that this algorithm provides a poor approximation for games with large alphabets.

4 Hardness of Approximating Label Cover and Unique Label Cover

Observation: If we can show that for a problem P, given an instance x, such that either $P(x) \geq \alpha$ or $P(x) < \beta$, it is **NP**-hard to distinguish between the two cases, then P is **NP**-hard to approximate within β/α.

In the mini-course we will show the following theorem about the hardness of approximating Label Cover:

THEOREM 6 (PROJECTION GAMES THEOREM)
*There is a constant $c \in (0,1)$ such that for $\epsilon = \epsilon(n) \geq \frac{1}{n^c}$, there is a $k = k(\epsilon)$, such that it is **NP**-hard to distinguish given a projection game with k labels on a graph of size n, whether the value is 1 or at most ϵ.*

REMARK 7

1. We will see a proof that gives $k(\epsilon) = 2^{\text{poly}(\frac{1}{\epsilon})}$ [M-Raz, 08], while it seems that the truth should be $k(\epsilon) = \Theta(1/\epsilon)$.
2. Parallel repetition gives a reduction from solving SAT on inputs of size N to Label-Cover on inputs of size $N^{\Theta(\log 1/\epsilon)}$ [Raz, 94]. We can interpret this as proving the above theorem for every *constant* ϵ, or as proving the above theorem for *quasi-**NP**-hardness*. Note that parallel repetition gives a much better $k(\epsilon) = \text{poly}(1/\epsilon)$.

For Unique Label Cover, we have the following conjecture:

CONJECTURE 8 (UNIQUE GAMES CONJECTURE, KHOT02) *For every constant $\epsilon > 0$, there is a $k = k(\epsilon)$ such that given a unique game with k labels, it is **NP**-hard to distinguish whether the value of the game is at least $1 - \epsilon$ or at most ϵ.*

Note that it was necessary to replace the 1 with $1 - \epsilon$ for unique games, since checking whether there is an assignment to a unique game that satifies all edges can be done efficiently.

The Implication of the Projection Games Theorem to Probabilistic Checking of Proofs We will show a reduction from checking whether a Boolean formula φ is satisfiable to approximating Label-Cover.

Interpret the labels to the vertices as a proof for the satisfiability of φ. The alphabet of the proof is $\Sigma_A \cup \Sigma_B$, and the length of the proof is the number of vertices. We can probabilistically verify the proof as follows: we pick an edge at random and check whether it is satisfied. Thus, we have only two queries to the proof. Moreover, by the projection, already after making the first query, we know exactly what should be the value for the other query!

We will show *Completeness*: if φ is satisfiable, then there exists a proof that is accepted with probability 1. We will also show *Soundness*: if φ is unsatisfiable, then for any proof, the probability of acceptance is at most ϵ.

This means that any mathematical statement, given as a Boolean formula φ, can be checked probabilistically with error tending to 0 by reading only two positions in the proof!

The Implications of the Projection Games Theorem to Hardness of Approximation The Projection Games Theorem is used to derive hardness of approximation results for other problems via a scheme by Bellare, Goldreich and Sudan (95). This paradigm has been very successful in proving optimal hardness results for problems such as 3SAT ($\frac{7}{8}+\delta$), 3LIN ($\frac{1}{2}+\delta$) [Håstad, 97] and SET-COVER ($(1-\delta)\ln n$) [Feige, 95]. For other problems like MAX-CUT ($\frac{16}{17}+\delta$) [Håstad, 97] and VERTEX COVER (≈ 1.36 [Dinur-Safra, 02]), the results were not optimal. Khot proposed using the Unique Games Conjecture instead of the Projection Games Theorem, and that led to optimal hardness results for many of these problems under the conjecture. For example, under the Unique Games Conjecture, we have hardness factor $2-\delta$ for VERTEX-COVER [Khot-Regev, 03] and hardness factor matching the Gomens-Williamson factor ≈ 0.878 for Max-Cut [KKMO, 04]. Raghavendra (08) showed that under the UGC, using the above scheme, we can get optimal hardness for all D-CSPs for constant D and constant alphabet.