EXERCISE 1 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

DMITRY GOUREVITCH

A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign ($\square$) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems. Also I will try to single out problems that are not directly related to algebraic geometry. For example sign (CA) denotes a problem (or a definition) from commutative algebra, (LA) stands for linear algebra, (Top) for topology.

Remark. In this assignment you can freely use the Nullstellensatz. Please specify every time you use it (for example put the sign (NS)). Check for yourself that you really need it for your argument to work.

(1) (a) Let A be a finitely generated k-algebra with no nilpotents. Consider the set $X = Mor_{k-alg}(A, k)$. Show that A embeds into the algebra $F(X)$ of k-valued functions on X and that the pair $(X; A)$ is an affine algebraic variety.
   (b) If X, Y are affine algebraic varieties, show that $Mor(X; Y) = Mor_{k-alg}(P(Y), P(X))$.

Definition 1 (CA)**.** *Let A be a ring (commutative, with 1). Given an element $f \in A$ we define the localization of A with respect to f to be the ring $A_f = A[t]/(ft-1)A[t]$*

(2) (a) Show that we can define the ring A_f as a set of fractions $\frac{a}{f^k}$ modulo the following equivalence relation: $\frac{a}{f^k}$ is equivalent to $\frac{b}{f^l}$ if for some N we have $f^N f^l a = f^N f^k b$.
   (b) Describe the kernel of the canonical morphism $i : A \to A_f$
   (c) In what cases the localized ring A_f is trivial, i.e. consists of one element 0 ?
   (d) Show that if A has no nilpotents (resp. no zero divisors), then A_f has no nilpotents (resp. no zero divisors).

(3) (P) Consider the subvariety $Y \subset A^2$ defined by one equation $x^2 - y^3 = 0$. Describe a morphism $\nu : A^1 \to Y$ which is bijective. Show that it is a homeomorphism, but not an isomorphism of algebraic varieties.

(4) (P) Let K be some field. Consider a system of polynomial equations $\Xi = (P_\alpha)$, where P_α are polynomials in variables $x_1, ..., x_n$. Suppose we know that there exists a field extension $K \subset L$ such that the system of equations Ξ has some solution in L , i.e. there exists a collection of elements $\xi_i \in L$ that satisfy all the equations in Ξ. Show that the system Ξ has a solution in the field $\overline{K}$ - the algebraic closure of K.

(5) (P) Let X be an affine algebraic variety. Denote by B the family of basic open subsets of X. Show that B forms a base of a topology and that the topology defined by B is the Zariski topology on X.

Definition 2 (Top)**.** *A topological space X is called quasicompact if any open covering $\{U\}_\alpha$ of X has finite subcovering.*

(6) (P) Show that any affine algebraic variety is quasicompact in Zariski topology.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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