EXERCISE 2 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

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(1) (P) Let X be an algebraic variety. Fix a finite open covering of X consisting of affine algebraic varieties B_i. Assume that all the intersections $B_i \cap B_j$ are affine (this will be true for most of interesting cases).

Show that a function on X is regular if its restriction to every subset B_i is polynomial. Show how using the covering $\{B_i\}$ one can give an explicit description of the space of regular functions $\mathcal{O}(X)$ as the kernel of a morphism

$$\phi : \bigoplus_i \mathcal{O}(B_i) \to \bigoplus_{i,j} \mathcal{O}(B_i \cap B_j).$$

(2) Fix a finite dimensional vector space V over k.
   (a) Let f be a coordinate function on V (i.e. a non-zero linear function). Consider the basic open subvariety $V_f \subset V$. Show that V_f is an affine algebraic variety and that it is isomorphic to $H \times k^\times$, where $H \subset V$ is the hyperplane given by equation $f(x) = 1$. Describe explicitly $\mathcal{O}(V_f)$.
   (b) (P) Denote by $V^\times$ the open algebraic subvariety $V^\times := V \setminus \{0\}$. Describe the algebra $\mathcal{O}(V^\times)$.
   Hint. Do this first for the case $\dim V = 1$, then $\dim V = 2$, then the general case.

The quotient construction. Let $(X; T(X); \mathcal{O}(X))$ be a space with a sheaf of functions. Let $p : X \to Y$ be an epimorphic map of sets. Show how in this case one can canonically de
ne on Y the structure of a space with sheaf of functions.

(3) Fix a finite dimensional vector space V over k and let $X = \mathbb{P}(V)$ be the set of one-dimensional subspaces of V (projective space). Consider the natural projection of sets $p : V^\times \to X$. Using the quotient construction define topology $T(X)$ and a sheaf of functions $\mathcal{O}(X)$ on the set X. Show that this is an algebraic variety. More precisely show that for any coordinate function f the image $p(V_f)$ is an open subset of X isomorphic to an affine space (as a space with a sheaf of functions). We will denote this variety by $\mathbb{P}(V)$. In case when V is the standard coordinate space with coordinates $(t_o; ...; t_n)$ this variety is usually denoted by $\mathbb{P}^n$.

(4) (P)
   (a) Describe the algebra $\mathcal{O}(\mathbb{P}^n)$ of global regular functions on $\mathbb{P}^n$.
   (b) Let X be an algebraic variety obtained from $\mathbb{P}^2$ by removing one point. Describe the algebra $\mathcal{O}(X)$ of global regular functions on X.

Definition 1 (Top). *Let X be a topological space. A subset $Y \subset X$ is called* locally closed *if it satisfies the following equivalent conditions*
   (a) Y is an intersection of an open and a closed subsets of X.
   (b) any point $x \in Y$ has an open neighborhood U such that $U \setminus Y$ is closed in U .
   (c) Y is open in its closure.

(5) Check that these conditions are equivalent.

(6) (P) Let X be an algebraic variety and $Z \subset X$ a locally closed subset. Show that Z has a canonical structure of an algebraic variety.

Definition 2 (LA). *Let H be a group. A* character *of H is a homomorphism of groups $H \to k^\times$, where $k^\times$ is the multiplicative group of k.*

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Suppose we fixed an action of a group H on a finite-dimensional k-vector space L. For any character χ of H we consider the eigensubspace

$$L_\chi := \{v \in L | hv = \chi(h)v \, \forall h \in H\}$$

(7) (P)(LA). Let a group H act on a space L. Fix a collection of pairwise distinct characters $\chi_1, ..., \chi_m$ of H. Suppose we have an equality in L of the form $v = \sum v_i$, where $v_i \in L_{\chi_i}$. Then every v_i can be written as a linear combination of vectors of the form hv, with $h \in H$. In particular show that if $v = 0$ then all vectors v_i are zeroes.

(8) (P) Let V be a finite-dimensional vector space over k and $X := \mathbb{P}(V)$ the corresponding projective space (see the problem 2). We would like to describe the alebraic structure of this space in more detail. Later we will use this description many times. Denote by A the algebra of polynomial functions on V . This is a graded algebra, i.e. $A = \bigoplus_{k \in \mathbb{Z}} A^k$ with $A^k A^l \subset A^{k+l}$ and $k \subset A^0$. Given a homogeneous polynomial $f \in A^k$ with $k > 0$ we consider the corresponding basic open subset $V_f \subset V$ and denote by X_f its image in the projective space X.
   (a) Show that the sets X_f form a basis of the Zariski topology on X.
   (b) Show that every subset X_f is an affine algebraic variety and the algebra of regular functions $\mathcal{O}(X_f)$ is isomorphic to the subalgebra A_f^0 of functions of degree zero in the graded algebra A_f.
   Hint. Use the previous problem.

(9) (*) Let V be an n-dimensional k-vector space. Fix a number $1 \leq l \leq n$ and denote by $Gr_l(V)$ the set of all subspaces $L \subset V$ of dimension l. Describe the natural structure of an algebraic variety on this set. This variety is called the Grassmann variety or the Grassmannian.
Hint. First understand that in the space of matrices $Mat(l \times n)$ the subset of matrices of any given rank r is an algebraic subvariety.

(10) Let A be a finitely generated k-algebra (commutative, with 1). Consider the set $M(A) := Mor_{k-alg}(A, k)$.
   (a) Describe Zariski topology on the set $M(A)$. Show that the set $M(A)$ has a natural structure of an affine algebraic variety. In particular describe the algebra of regular functions $\mathcal{O}(M(A))$.
   (b) Given a morphism of k-algebras $\phi : B \to A$ show that the corresponding map $\phi^* M(A) \to M(B)$ is a morphism of affine algebraic varieties.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html