EXERCISE 3 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

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In this exercise A, B denote rings (commutative, with 1), and k an algebraically closed field.

(1) (a) Let $m \subset A$ be an ideal. Prove that m is maximal if and only if A/m is a field.
   (b) (P) Suppose that A is a finitely generated k-algebra and m be a maximal ideal. Show that $A/m \simeq k$.
   Hint: use NSS.

(2) Let $\phi : A \to B$ be a morphism of rings.
   (a) Show that the inverse image of a prime ideal is a prime ideal.
   (b) Give an example in which the inverse image of a maximal ideal is not a maximal ideal.
   (c) Prove that if A, B are finitely-generated k-algebras then the inverse image of a maximal ideal is always a maximal ideal.

(3) (P) Let A be a unique factorization domain.
   (a) For a polynomial $p = \sum_{i=0}^{d} a_i x^i$ define $c(p) := gcd(a_0, ..., a_d)$. Show that $c(pq) = c(p)c(q)$
   (b) Let $K := (A \setminus 0)^{-1}A$ be the field of fractions of A, and let $r \in A[x]$ be an irreducible monic polynomial. Show that r is irreducible also in $K[x]$.
   (c) Show that $A[x]$ is a unique factorization domain. Deduce by induction that $k[x_1, ..., x_n]$ is a unique factorization domain.

(4) (P) Let $Nil(A)$ denote the set of all nilpotent elements of A, and $A^\times$ the group of invertible elements. Show that:
   (a) $Nil(A)$ is an ideal, and the ring $B := A/Nil(A)$ has no nilpotents.
   (b) If $a \in Nil(A)$ then $1 + a \in A^\times$.
   (c) $A^\times/(1 + Nil(A)) \simeq B^\times$
   (d) (*) $Nil(A)$ is the intersection of all prime ideals of A. You can assume that A is Noetherian.

(5) (P) Show that a polynomial $p \in A[x]$ is invertible if and only if the free coefficient is invertible and other coefficients are nilpotent.

(6) (a) Prove that a Noetherian ring has a maximal ideal.
   (b) (∇) Using Zorn's lemma, prove that any ring has a maximal ideal.

(7) (P) Let M be a finitely generated A-module and $T : M \to M$ be an endomorphism.
   (a) Show that if T is epimorphic (onto) then it is invertible. Moreover show that it is invertible on any T-invariant A-submodule $L \subset M$.
   (b) Give an example of T that is monomorphic (1-1) but not invertible.

(8) (P) Let X be a (reducible) affine algebraic variety, and $Z_1, Z_2 \subset X$ closed subsets such that $X = Z_1 \cup Z_2$. Let f be a function on X such that the restrictions f_{Z_1} and f_{Z_2} are regular (polynomial) functions. Then f is a regular function on X.
   Hint: use the lemma saying that

$$(A/I \oplus A/J)/(\Delta(A/(I \cap J)) \simeq A/(I + J)$$

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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