EXERCISE 4 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

DMITRY GOUREVITCH

(1) (P) Let A be a unique factorization domain.
   (a) For a polynomial $p = \sum_{i=0}^{d} a_i x^i$ define $c(p) := gcd(a_0, ..., a_d)$. Show that $c(pq) = c(p)c(q)$
   (b) Let $K := (A \setminus 0)^{-1}A$ be the field of fractions of A, and let $r \in A[x]$ be an irreducible monic polynomial. Show that r is irreducible also in $K[x]$.
   (c) Show that $A[x]$ is a unique factorization domain. Deduce by induction that $k[x_1, ..., x_n]$ is a unique factorization domain.

(2) (P) Show that any affine algebraic variety is a union of finitely many irreducible components (i.e. irreducible closed subvarieties).

(3) (P) Let X be a (reducible) affine algebraic variety, and $Z_1, Z_2 \subset X$ closed subsets such that $X = Z_1 \cup Z_2$. Let f be a function on X such that the restriction $f|_{Z_1}$ is a regular (polynomial) function and $f|_{Z_2} = 0$.
   (i) Show that some power of f is a regular function on X. Hint: use the lemma saying that
$$(A/I \oplus A/J)/(\Delta(A/(I \cap J)) \simeq A/(I + J)$$
   (ii) Give an example of X and f such that f is not a regular function on X.
   (iii) Give an example of X and a function g on X such that $g|_{Z_1}$ and $g|_{Z_2}$ are regular functions, but no power of g is a regular function on X.

(4) (P) (Nakayama lemma). Let J be an ideal in a commutative ring A. Let M be a finitely generated A-module such that $JM = M$. Then there exists an element $j \in J$ such that $(1 - j)M = 0$. This implies $JL = L$ for any submodule $L \subset M$. Hint: use induction on the number of generators by picking $x \in M$ and using induction hypothesis for M/Ax.

(5) (P) Prove the following generalization of the central lemma in the proof of NSS:
   (a) For any (endo)morphism $T : M \to M$ there exists a monic polynomial $Q \in K[t]$ such that $M/Q(T)M \neq 0$.
   (b) If the field K is algebraically closed then there exists a constant $\lambda \in K$ such that the module $M = (T - \lambda)M \neq 0$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

Date: July 29, 2016.