EXERCISE 5 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

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(1) (P) Prove the following properties of Krull dimension (defined using maximal chain of irreducible subvarieties).
(a) If \( X_i \) are (locally closed) subvarieties of \( X \) and \( X = \bigcup_{i=1}^{k} X_i \) then \( \dim X = \max \dim X_i \).
(b) If \( \nu : X \to Y \) is dominant (i.e. has dense image) then \( \dim X \geq \dim Y \).
(c) If \( \nu : X \to Y \) is finite then \( \dim X \leq \dim Y \).
(d) \( \dim \mathbb{A}^n = n \).
(e) For any \( X \), \( \dim X \) is finite.
(f) If \( f \) is a non-constant polynomial in \( n \) variables then \( \dim \text{Zeroes}(f) = n - 1 \).

(2) (P) Let \( d \) be a "function" from algebraic varieties to natural numbers (including zero) that satisfies:
(a) If \( X_i \) are (locally closed) subvarieties of \( X \) and \( X = \bigcup_{i=1}^{k} X_i \) then \( d(X) = \max d(X_i) \).
(b) If \( \nu : X \to Y \) is a finite epimorphism (=onto map) then \( d(X) = d(Y) \).
(c) \( d(\mathbb{A}^n) = n \) for any \( n \).
Show that \( d(X) = \dim X \) for any \( X \).

(3) (*) Let \( X \) be an affine irreducible variety and \( L \) be the field of rational functions on \( X \). Show that \( \dim X \) equals the transcendence degree of \( L \) over \( k \).

(4) (P) Prove the following generalization of the central lemma in our proof of NSS:
(a) For any (endo)morphism \( T : M \to M \) there exists a monic polynomial \( Q \in K[t] \) such that \( M/Q(T)M \neq 0 \).
(b) If the field \( K \) is algebraically closed then there exists a constant \( \lambda \in K \) such that the module \( M = (T - \lambda)M \neq 0 \).

(5) (P) Let \( A \) be a (commutative) ring.
(a) Show that if \( A^r \simeq A^s \) then \( r = s \). Hint: consider quotient by a maximal ideal.
(b) Suppose that \( A \) has infinitely many maximal ideals. Let \( m_1, \ldots, m_k \subset A \) be some maximal ideals. Suppose that we have an isomorphism of modules \( A^r \oplus (\bigoplus A/m_i^{l_i}) \simeq A^s \oplus (\bigoplus A/m_i^{n_i}) \).
Show that \( r = s \).
(c) Suppose that \( A \) is a principal ideal domain. Let \( p_1, \ldots, p_k \in A \) be prime elements and suppose that
\[
A^r \oplus (\bigoplus A/p_i^{l_i}) \simeq A^s \oplus (\bigoplus A/p_i^{n_i}).
\]
Then \( r = s \) and \( l_i = n_i \) for all \( i \). This exercise complements the theorem from the lecture and shows that the decomposition of any module in the above form is unique. Hint: prove this carefully by induction, quotienting by different powers of different prime elements.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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