EXERCISE 5 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

DMITRY GOUREVITCH

(1) (P) Prove the following properties of Krull dimension (defined using maximal chain of irreducible subvarieties).
   (a) If X_i are (locally closed) subvarieties of X and $X = \cup_{i=1}^k X_i$ then $\dim X = \max \dim X_i$.
   (b) If $\nu : X \to Y$ is dominant (i.e. has dense image) then $\dim X \geq \dim Y$
   (c) If $\nu : X \to Y$ is finite then $\dim X \leq \dim Y$.
   (d) $\dim \mathbb{A}^n = n$
   (e) For any X, $\dim X$ is finite
   (f) If f is a non-constant polynomial in n variables then $\dim Zeroes(f) = n - 1$

(2) (P) Let d be a "function" from algebraic varieties to natural numbers (including zero) that satisfies:
   (a) If X_i are (locally closed) subvarieties of X and $X = \cup_{i=1}^k X_i$ then $d(X) = \max d(X_i)$.
   (b) If $\nu : X \to Y$ is a finite epimorphism (=onto map) then $d(X) = d(Y)$.
   (c) $d(\mathbb{A}^n) = n$ for any n.

   Show that $d(X) = \dim X$ for any X.

(3) (*) Let X be an affine irreducible variety and L be the field of rational functions on X. Show that $\dim X$ equals the transcendence degree of L over k.

(4) (P) Prove the following generalization of the central lemma in our proof of NSS:
   (a) For any (endo)morphism $T : M \to M$ there exists a monic polynomial $Q \in K[t]$ such that $M/Q(T)M \neq 0$.
   (b) If the field K is algebraically closed then there exists a constant $\lambda \in K$ such that the module $M = (T - \lambda)M \neq 0$.

(5) (P) Let A be a (commutative) ring.
   (a) Show that if $A^r \simeq A^s$ then $r = s$. Hint: consider quotient by a maximal ideal.
   (b) Suppose that A has infinitely many maximal ideals. Let $m_1, \ldots, m_k \subset A$ be some maximal ideals. Suppose that we have an isomorphism of modules $A^r \oplus (\bigoplus A/m_i^{l_i}) \simeq A^s \oplus (\bigoplus A/m_i^{n_i})$. Show that $r = s$.
   (c) Suppose that A is a principal ideal domain. Let $p_1, \ldots, p_k \in A$ be prime elements and suppose that
$$A^r \oplus (\bigoplus A/p_i^{l_i}) \simeq A^s \oplus (\bigoplus A/p_i^{n_i}).$$
   Then $r = s$ and $l_i = n_i$ for all i. This exercise complements the theorem from the lecture and shows that the decomposition of any module in the above form is unique. Hint: prove this carefully by induction, quotienting by different powers of different prime elements.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

Date: December 3, 2012.