EXERCISE 6 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

DMITRY GOUREVITCH

(1) (P) Compute the following tensor products
   (a) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$
   (b) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$
(2) (P) Let M, N and T be modules over a ring R.
   (a) For any map $\phi : M \to N$ define a natural map $\phi_T : T \otimes_R M \to T \otimes_R N$.
   (b) Show that if ϕ is onto then ϕ_T is onto
   (c) Give an example when ϕ is 1-1 but ϕ_T is zero.
(3) Finish the details in the definition of product of function spaces.
(4) Finish the proof of the theorem saying that product of (affine) varieties is an (affine) variety.
(5) (P) Show that $\mathbb{A}^1 \times \cdots \times \mathbb{A}^1 \simeq \mathbb{A}^n$.
(6) (P) Show that if $X = Z(I) \subset \mathbb{A}^n, Y = Z(J) \subset \mathbb{A}^k$ then $X \times Y = Z(< I, J >) \subset \mathbb{A}^{n+k}$.
(7) (P) Show that if X, Y are varieties then $\dim(X \times Y) = \dim X + \dim Y$ and if X, Y are irreducible then so is $X \times Y$.

Definition 1. *A variety is called* projective *if it can be embedded into a projective space as a closed subset, and* quasi-projective *if it can be embedded into a projective space as a locally-closed subset.*

Note that affine varieties are not projective, but are quasi-projective. In fact, all varieties we considered up to now are quasi-projective.

(8) (P) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety.
Hint. Use the embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ given by $(v) \times (w) \to (v \otimes w)$.
Deduce that product of (quasi-) projective varieties is (quasi-) projective.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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