# EXERCISE 7 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

DMITRY GOUREVITCH

(1) (P) Let $X, Y$ be algebraic varieties, where $Y$ is separated, and $U \subset X$ be a open dense subset.
   (a) Let $f, g : X \to Y$ be morphisms that agree on $U$. Show that $f = g$.
   (b) Let $h : U \to Y$ be a morphism. Show that there exists a maximum open subset $O \subset X$ to which $h$ extends as a morphism.

(2) (P) Let $X$ be a complete algebraic variety.
   (a) Show that any closed subvariety $Z \subset X$ is complete.
   (b) Let $f : X \to Y$ be a morphism of algebraic varieties. Show that the image $f(X)$ is a closed complete subvariety of $Y$.
   (c) Show that a quasi-affine complete variety is a finite set.

(3) Let $f_i(x_0, \dots, x_n), 0 \leq i \leq N = \binom{n+d}{n} - 1$ be the set of all monomials in $k[x_0, \dots, x_n]$ of degree $d$, i. e. of the monomials of the form $x_0^{i_0} \cdots x_n^{i_n}$ with $i_0 + \dots + i_n = d$. Consider the map $F : \mathbb{P}^n \to \mathbb{P}^N, (x_0 : \dots : x_n) \mapsto (f_0(x_0, \dots, x_n) : \dots : f_N(x_0, \dots, x_n))$. The morphism $F$ is called the degree-d Veronese embedding. Its importance lies in the fact that degree-d polynomials in the coordinates of $\mathbb{P}^n$ are translated into linear polynomials when viewing $P^n$ as a subvariety of $\mathbb{P}^N$.

   Show that $F$ is a closed imbedding, i.e. isomorphism to its closed image.

(4) (P) Let $X \subset$P$^n$ be a projective variety, and let $f \in k[x_0, \dots, x_n]$ be a nonconstant homogeneous polynomial. Show that $X \setminus Zeroes(f)$ is an affine variety.

   **Hint.** Use the Veronese embedding.

*URL*: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

*Date*: December 24, 2012.