EXERCISE 7.5 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

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(1) Let $f_i(x_0,\dots,x_n), 0 \leq i \leq N = \binom{n+d}{n} - 1$ be the set of all monomials in $k[x_0,\dots,x_n]$ of degree d, i. e. of the monomials of the form $x_0^{i_0}\cdots x_n^{i_n}$ with $i_0+\cdots+i_n = d$. Consider the map $F : \mathbb{P}^n \to \mathbb{P}^N, (x_0 : \cdots : x_n) \mapsto (f_0(x_0,\dots,x_n) : \cdots : f_N(x_0,\dots,x_n))$. The morphism F is called the degree-d Veronese embedding. Its importance lies in the fact that degree-d polynomials in the coordinates of $\mathbb{P}^n$ are translated into linear polynomials when viewing P^n as a subvariety of $\mathbb{P}^N$.

Show that F is a closed imbedding, i.e. isomorphism to its closed image.

(2) (P) Let $X \subset \mathbb{P}^n$ be a projective variety, and let $f \in k[x_0,\dots,x_n]$ be a nonconstant homogeneous polynomial. Show that $X \setminus Zeroes(f)$ is an affine variety.

Hint. Use the Veronese embedding.

(3) (a) Let V be an n-dimensional vector space over k. Compute the dimension of the Grassmannian manifold $Gr_l(V)$ - the variety of all l-dimensional subspaces of V.

Hint. Let $GL(V)$ denote the group of all isomorphisms of V to itself. Fix any l-dimensional subspace $W \subset V$ and consider the onto map $GL(V) \to Gr_l(V)$ given by action on W. Use the lemma we had in class on dimensions of fibers.

(b) Consider the subset $M_r(m,n)$ of the space $M(m,n)$ of $m \times n$ matrices consisting of all matrices of rank r. Show that this is an algebraic variety and compute its dimension.

(c) Compute dimension of the set of all quadratic hypersurfaces in $\mathbb{P}^6$. (Quadratic hypersurface is a hypersurface given by equation of degree 2).

(4) Let $f : X \to Y$ be a morphism of varieties, and let $Z \to X$ be a closed subset. Assume that the fiber $f^{-1}(P) \setminus Z$ is irreducible and of the same dimension for all points $P \in Y$. Prove that then Z is irreducible too. (This is a quite useful criterion to check the irreducibility of closed subsets.)

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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