EXERCISE 9 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY

DMITRY GOUREVITCH

(1) (P) Let X be a variety, and let $Y \subset X$ be a closed subset. For every element in an open affine cover U_i of X, let $V_i := U_i \cap Y$ and let $\tilde{U_i}$ be the blow-up of U_i at V_i. Show that the spaces $\tilde{U_i}$ can be glued together to give a variety $\tilde{X}$. This variety is then called the blow-up of X at Y.

(2) (P) A quadric in $\mathbb{P}^n$ is a projective variety in $\mathbb{P}^n$ that can be given as the zero locus of a quadratic polynomial. Show that every quadric in $\mathbb{P}^n$ is birational to $\mathbb{P}^{n-1}$.

(3) (P) Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. Show that the coordinate ring $\mathcal{O}(C_P X)$ of the tangent cone to X at P is equal to $\oplus_{l \geq 0} I(P)^l/(I(P)^{l+1})$, where $I(P)$ is the ideal of P in $\mathcal{O}(X)$.

(4) (P) Let $X \subset \mathbb{A}^n$ be an irreducible affine variety, and let $Y_1, Y_2 \subset X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of X at the (possibly nonradical) ideal $I(Y_1) + I(Y_2)$. Then the strict transforms of Y_1 and Y_2 on $\tilde{X}$ are disjoint.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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