EXERCISE 1 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY II

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(1) (P) Let X be an irreducible algebraic variety, $x \in X$.
   (a) Show that $\mathcal{O}_{X,x}$ is Noetherian.
   (b) If $X = \text{Spec}A, m_x \subset A$, show that $\mathcal{O}_{X,x} = A_{m_x} \equiv S^{-1}A$, where $S = A \setminus m_x$.
   (c) Is it possible that $\mathcal{O}_{X,x_1} \subset \mathcal{O}_{X,x_2} \subset K(x)$ but $m_{x_1} \not\subset m_{x_2}$
(2) (P) Show that:
   (a) Integral closure commutes with localization.
   (b) For any irreducible variety Y and any finite extension $L \supset K(Y)$ $\exists!$ normal irreducible variety X and a finite surjective map $\nu : X \to Y$ s.t. $\nu^* : K(Y) \hookrightarrow K(X)$ is the imbedding $K(Y) \subset L$
(3) (P) Let $A = K[x, y]/ < x^2 - y^3 >$ and $C = SpecA$.
   (a) Show that neither A nor its localization at $(0, 0)$ are integrally closed.
   (b) Show that the localization at any other point is integrally closed.
(4) Let R be Discrete Valuation Ring and $\pi \in R$ be irreducible, then any $r \in R$ has a unique representation as $u \cdot \pi^n, u \in R^*$ and $n \geq 0$. n is called the valuation of r. Show that it satisfies:
   (a) $val(ab) = val(a) + val(b)$
   (b) $val(a + b) \geq \min(val(a), val(b))$
(5) Let X be algebraic variety, U be a constructable subset, $x \in X$. Show that $x \in \overline{U} \iff \exists$ a smooth curve C and $\nu : C \to X$ s.t. $x \in \nu(C)$ and $\#\nu(C) \cap U = \infty$.
(6) (P) If C is a smooth curve, $k[C] \neq k$ then C is affine.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

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