EXERCISE 2 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY II - INTRODUCTION TO RIEMANN-ROCH THEOREM

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Let C be a smooth projective curve. Show that

(1) $f \in L(D) \iff D + (f) \geq 0$.

(2) Let $f \in k(C)$, and let $\tilde{f}$ be the corresponding morphism $C \to \mathbb{P}^1$. Then $(f) = (\tilde{f})^{-1}(\infty - 0)$.

(3) (P) For a non-constant morphism $\nu : C \to \gamma$ and any $D \in Div(\gamma)$, $\deg(\nu^{-1}(D)) = \deg \nu \cdot \deg(D)$.

(4) (P) If $S \subset C$ is a finite subset, and $D \leq D' \in Div(C)$ then $dim(L^S(D')/L^S(D)) = deg^S(D' - D)$, where $deg^S(D) = \sum_{P \in S} n_P$ and $L^S(D) = \{f \in k(C) | ord_P(f) \geq n_P \forall P \in S\}$.

(5) (P) The following are equivalent:
 (a) $C \simeq \mathbb{P}^1$
 (b) For some $P \in C, l(P) > 1$

 Hint: use Proposition 2 from the lecture notes (that unfortunately was missing from the lecture).

(6) (P) Let $C = \mathbb{P}^1$, and let $P \in C$. Then $l(nP) = n + 1$ and the genus of $\mathbb{P}^1$ is zero.

(7) (P) Using Riemann's theorem, show that there exists an integer N s.t. for all divisors D of degree $> N$, $l(D) = deg(D) + 1 - g$.

(8) (P) Using Riemann-Roch theorem, show that
 (a) $l(W) = g$, where W is the canonical divisor.
 (b) If $deg(D) \geq 2g - 1$, then $l(D) = deg(D) + 1 - g$.
 (c) If $deg(D) \geq 2g$, then $l(D - P) = l(D) - 1$ for all $P \in C$.
 (d) (Cliffords Theorem). If $l(D) > 0$, and $l(W - D) > 0$ then

$$l(D) \leq \frac{deg(D)}{2} + 1.$$

(9) (P) Let D be any divisor, $P \in C$. Then $l(W - D - P) \neq l(W - D)$ if and only if $l(D + P) = l(D)$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

Date: May 8, 2013.