EXERCISE 4 IN COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY II

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(1) Let X be a topological space.
  (a) What is a monomorphism and an epimorphism in PreSh(X)?
  (b) (P)* What is a monomorphism and an epimorphism in Sh(X)?

(2) (P)* Show that a category is abelian if and only if it satisfies the following:
  (a) $\exists$ 0 object.
  (b) $\exists$ 0 direct sums and direct products.
  (c) Any monomorphism is the kernel of some morphism, and any epimorphism is the cokernel of some morphism.

(3) Show that the localization functor is exact, but does not commute with infinite direct products.

(4) Show that If M is a finitely generated A- module then $\mathrm{Hom}(M, \oplus N_\alpha) = \oplus \mathrm{Hom}(M, N_\alpha)$.

(5) (P) Let M be a finitely generated module over a ring A. Show that M is projective if and only if it is locally free, i.e. M_m is free over A_m for any maximal ideal m of A. Hint: first assume that A is local.

(6) (P) Let M be a module over a ring A. Show that the following are equivalent.
  (a) M is flat.
  (b) For any embedding $K \subset L$, the natural map $K \otimes_A M \to L \otimes_A M$ is an embedding
  (c) For any ideal $I \subset A$, the natural map $I \otimes_A M \to M$ is an embedding.
  (d) For any pair of tuples $m \in M^k$, $r \in A^k$ with $r^t m := \sum r_i m_i = 0$ there exists a matrix $X \in Mat(A, k \times k)$ and a tuple $n \in M^k$ s.t. $An = m$ and $r^t A = 0$.
  (e) There exists a directed system of finitely generated free modules F_α with $\lim_\to F_\alpha = M$.

(7) (P) Any finitely presented flat module is projective.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgGeo.html

Date: May 19, 2013.