Automorphic representations.

Problem assignment 1. 06-11-2018

Consider the group $G = SL(2, \mathbf{R})$ and let $K = SO(2)$ be its maximal compact subgroup.

Consider the standard action of the group G on the upper half-plane $\mathbf{H}$ by fractional linear transformations. This action is transitive and the stationary subgroup of the point $x_0 = i \in \mathbf{H}$ is equal to K. In this way we always identify $\mathbf{H}$ with the quotient space G/K.

Let $\mathcal{H} = \mathcal{H}(G)$ denote the space of smooth measures on G with compact support. This space is an associative algebra (without 1) with respect to convolution. Any representation (π, G, V) extends to a representation of this algebra via formula $\pi(h) := \int_G \pi(g)h(g)$.

Since $\mathbf{H} = G/K$ we can identify the space of functions $F(\mathbf{H})$ with the space $F(G/K) = F(G)^K$ – the space of right K-invariant functions on G.

Consider the subalgebra $\mathcal{H}_K = \mathcal{H}(G//K) \subset \mathcal{H}$ of all K bi-invariant measures. It acts on the space $F(\mathbf{H}) = F(G/K)$ via $\rho(h)(f) = f * h$. (This is a right action, but we will see that the algebra $\mathcal{H}_K$ is commutative, so it does not matter). This action clearly commutes with the action of G on $F(\mathbf{H})$.

1. The algebra $\mathcal{H}_K$ is commutative.

Hint. Use Gelfand trick.

(i) Consider the transposition anti-involution $g \mapsto g^t$.

(ii) Show that it defined an anti-automorphism $t : \mathcal{H} \to \mathcal{H}$

(iii) Show that the restriction of the morphism t to the subalgebra $\mathcal{H}_K$ is identity.

The upper half plane $\mathbf{H}$ we consider with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. This metric is G-invariant (and has constant curvature -1). Corresponding Laplace-Beltrami operator Δ equals $y^2(\partial_x^2 + \partial_y^2)$. This operator is not positive, so we denote $D = -\Delta$. This is a non-negative G-equivariant operator on $\mathbf{H}$.

2. Show that operator D commutes with the action of the algebra $\mathcal{H}_K$.

We are interested in eigenfunctions of the operator D on the space H and on its quotients.

3. For a fixed $\lambda \in \mathbf{C}$ let us denote by F_λ the space of eigen-functions of D with eigenvalue λ. This space is G-invariant and $\mathcal{H}_K$-invariant.

Show that there exists **unique** K-invariant function $\Psi_\lambda \in F_\lambda$. such that $\Psi_\lambda(x_0) = 1$. It is called **spherical function**.

Show that for any $h \in \mathcal{H}_K$ we have $\rho(h)(\Psi_\lambda) = C\Psi_\lambda$. This constant C is uniquely defined; it is denoted $\hat{h}(\lambda)$. The function $\lambda \mapsto \hat{h}(\lambda)$ is called the **Selberg transform** of the element h. Show that this is a holomorphic function of λ.

Fix a discrete subgroup $\Gamma \subset G$ and consider the quotient space $X = \Gamma\backslash G$. It is called the **automorphic space**. Our main object of study will be the representation $(\Pi, G, F(X))$ of the group G on the space of functions on the automorphic space space X. Important tool is to study the corresponding representation Π of the Hecke algebra $\mathcal{H}$.

For starters we consider the space $F(X)^K$ of K-invariant functions. We identify it with the space $F(Y)$ of functions on the space $Y = X/K = \Gamma\backslash G/K = \Gamma\backslash\mathbf{H}$.

4. (i) Show that for $h \in \mathcal{H}$ the operator $\Pi(h) : F(X) \to F(X)$ is given by a smooth kernel function $K = K_h$, where $K(x, y) := \sum_\Gamma h(x^{-1}\gamma y)$.

(ii) Prove similar statement for the action of the element $h \in \mathcal{H}_K$ on the space $F(Y)$.

(iii) deduce from this that if the space X is compact, then the operator $\Pi(h)$ is compact, of trace class and its trace equals $\int_X K(x,x)dg$.

We have two actions on the space $F(Y)$ – action of the algebra $\mathcal{H}_K$ and action of the opearator Δ. From what we have seen they carry essentially the same information. Namely we have

5. Lemma. A function $f \in F(Y)$ is an eigenfunction of the operator Δ iff it is an eigenfunction of the algebra $\mathcal{H}_K$. The eigenvalue λ for operator Δ corresponds to the eigencharacter of the algebra $\mathcal{H}_K$ given by $h \mapsto \hat{h}(\lambda)$.

6. Suppose the space Y is compact. Show that there exists an orthonormal basis $f_0.f_1,..$ of the space $F(Y)$ such that f_i is an eigenfunction of the operator Δ with eigenvalue λ_i. We will number them so that $\lambda_0 < \lambda_1 \leq \lambda_2 \leq ....$ Collection of the eigenvalues λ_i we will call the automorphic spectrum of the space Y.

Technically it is often easier to work with integral operators than with differential ones.

Let E be a Hilbert space (we always assume that it is separable). Let $\mathcal{A}$ be some *-algebra and $\pi : A \to Op(E)$ be a star representation (by bounded operators in E).

Definition. We say that the representation π of A on E is completely reducible if E can be written as a Hilbert direct some of subspaces E_i that are A-invariant and the action of the algebra A on every space E_i is non-zero and irreducible.

7. (i) Show that if π is completely reducible then its restriction to any invariant subspace $E' \subset E$ is also completely reducible.

(ii) Suppose we have a system of A-invariant subspaces $E_i \subset E$ that are completely reducible. Show that then the closure E' of the sum of these subspaces is also completely reducible. Here we do not assume that the spaces E_i are orthogonal.

We would like to show that if the image $\pi(A)$ contains many compact operators, then the representation π is completely reducible.

8. For any subspace $F \subset E$ let us denote by E_F the closure of the space $\pi(A)(F)$ generated from F by the action of A. Prove the following

Proposition. Suppose we have a collection C of elements $a \in A$ such that

(i) All operators $\pi(a)$ for $a \in C$ are compact.

(ii) The subspace $F \subset E$ generated by images of the operators $\pi(a)$ for $a \in C$ is large, i.e. $E_F = E$.

Then the action of A on E is completely reducible.

Hints. (i) Can assume that A is a *-subalgebra of $Op(E)$ and it is closed in norm topology.

(ii) Suppose that the algebra $A \subset Op(E)$ has an element e that is an orthogonal projection on some subspace F. Consider the subalgebra $B = eAe \subset A$. Show that there is a natural bijection between (closed) A-invariant subspaces in E_F and (closed) B-invariant subspaces in F. This bijection preserves the inclusion relation and orthogonality.

(iii) Show that if in (ii) the space F is finite dimensional then the representation π on the space E_F is completely reducible.

(iv) Suppose the algebra A contains a compact self-adjoint operator X. For a real number $\mu \neq 0$ consider the space F of eigenvectors of X with eigenvalue μ. Show that this space is finite dimensional and that the orthogonal projector $e = e_F$ on this space also lies in the algebra A (here we use that A is norm closed in $Op(E)$).