Automorphic representations.

Problem assignment 1. 06-11-2018

Consider the group $G = \text{SL}(2, \mathbb{R})$ and let $K = \text{SO}(2)$ be its maximal compact subgroup.

Consider the standard action of the group $G$ on the upper half-plane $H$ by fractional linear transformations. This action is transitive and the stationary subgroup of the point $x_0 = i \in H$ is equal to $K$. In this way we always identify $H$ with the quotient space $G/K$.

Let $\mathcal{H} = \mathcal{H}(G)$ denote the space of smooth measures on $G$ with compact support. This space is an associative algebra (without 1) with respect to convolution. Any representation $\pi(G,V)$ extends to a representation of this algebra via formula $\pi(h) := \int_G \pi(g)h(g)$.

Since $H = G/K$ we can identify the space of functions $F(H)$ with the space $F(G/K) = F(G)^K$ – the space of right $K$-invariant functions on $G$.

Consider the subalgebra $\mathcal{H}_K = \mathcal{H}(G//K) \subset \mathcal{H}$ of all $K$ bi-invariant measures. It acts on the space $F(H) = F(G/K)$ via $\rho(h)(f) = f \ast h$. (This is a right action, but we will see that the algebra $\mathcal{H}_K$ is commutative, so it does not matter). This action clearly commutes with the action of $G$ on $F(H)$.

1. The algebra $\mathcal{H}_K$ is commutative.

   **Hint.** Use Gelfand trick.

   (i) Consider the transposition anti-involution $g \mapsto g'$.

   (ii) Show that it defined an anti-automorphism $t : \mathcal{H} \to \mathcal{H}$

   (iii) Show that the restriction of the morphism $t$ to the subalgebra $\mathcal{H}_K$ is identity.

   The upper half plane $H$ we consider with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. This metric is $G$-invariant (and has constant curvature $-1$). Corresponding Laplace-Beltrami operator $\Delta$ equals $y^2(\partial_x^2 + \partial_y^2)$. This operator is not positive, so we denote $D = -\Delta$. This is a non-negative $G$-equivariant operator on $H$.

2. Show that operator $D$ commutes with the action of the algebra $\mathcal{H}_K$.

   We are interested in eigenfunctions of the operator $D$ on the space $H$ and on its quotients.

3. For a fixed $\lambda \in \mathbb{C}$ let us denote by $F_\lambda$ the space of eigen-functions of $D$ with eigenvalue $\lambda$. This space is $G$-invariant and $\mathcal{H}_K$-invariant.

   Show that there exists unique $K$-invariant function $\Psi_\lambda \in F_\lambda$ such that $\Psi_\lambda(x_0) = 1$. It is called the **spherical function**.

   Show that for any $h \in \mathcal{H}_K$ we have $\rho(h)(\Psi_\lambda) = C\Psi_\lambda$. This constant $C$ is uniquely defined; it is denoted $\hat{h}(\lambda)$. The function $\lambda \mapsto \hat{h}(\lambda)$ is called the **Selberg transform** of the element $h$. Show that this is a holomorphic function of $\lambda$.

   Fix a discrete subgroup $\Gamma \subset G$ and consider the quotient space $X = \Gamma\backslash G$. It is called the **automorphic space**. Our main object of study will be the representation $(\Pi, G, F(X))$ of the group $G$ on the space of functions on the automorphic space space $X$. Important tool is to study the corresponding representation $\Pi$ of the Hecke algebra $\mathcal{H}$.

4. (i) Show that for $h \in \mathcal{H}$ the operator $\Pi(h) : F(X) \to F(X)$ is given by a smooth kernel function $K = K_h$, where $K(x,y) := \sum_{\gamma} \hat{h}(x^{-1}\gamma y)$.
(ii) Prove similar statement for the action of the element $h \in \mathcal{H}_K$ on the space $F(Y)$.

(iii) deduce from this that if the space $X$ is compact, then the operator $\Pi(h)$ is compact, of trace class and its trace equals $\int_X K(x,x)dg$.

We have two actions on the space $F(Y)$ – action of the algebra $\mathcal{H}_K$ and action of the opearator $\Delta$. From what we have seen they carry essentially the same information. Namely we have

**5. Lemma.** A function $f \in F(Y)$ is an eigenfunction of the operator $\Delta$ iff it is an eigenfunction of the algebra $\mathcal{H}_K$. The eigenvalue $\lambda$ for operator $\Delta$ corresponds to the eigencharacter of the algebra $\mathcal{H}_K$ given by $h \mapsto \hat{h}(\lambda)$.

**6.** Suppose the space $Y$ is compact. Show that there exists an orthonormal basis $f_0, f_1, \ldots$ of the space $F(Y)$ such that $f_i$ is an eigenfunction of the operator $\Delta$ with eigenvalue $\lambda_i$. We will number them so that $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Collection of the eigenvalues $\lambda_i$ we will call the automorphic spectrum of the space $Y$.

Technically it is often easier to work with integral operators than with differential ones.

Let $E$ be a Hilbert space (we always assume that it is separable). Let $\mathcal{A}$ be some *-algebra and $\pi : \mathcal{A} \to Op(E)$ be a star representation (by bounded operators in $E$).

**Definition.** We say that the representation $\pi$ of $\mathcal{A}$ on $E$ is completely reducible if $E$ can be written as a Hilbert direct sum of subspaces $E_i$ that are $\mathcal{A}$-invariant and the action of the algebra $\mathcal{A}$ on every space $E_i$ is non-zero and irreducible.

**7.** (i) Show that if $\pi$ is completely reducible then its restriction to any invariant subspace $E' \subset E$ is also completely reducible.

(ii) Suppose we have a system of $\mathcal{A}$-invariant subspaces $E_i \subset E$ that are completely reducible. Show that then the closure $E'$ of the sum of these subspaces is also completely reducible. Here we do not assume that the spaces $E_i$ are orthogonal.

We would like to show that if the image $\pi(\mathcal{A})$ contains many compact operators, then the representation $\pi$ is completely reducible.

**8.** For any subspace $F \subset E$ let us denote by $E_F$ the closure of the space $\pi(\mathcal{A})(F)$ generated from $F$ by the action of $\mathcal{A}$. Prove the following

**Proposition.** Suppose we have a collection $C$ of elements $a \in \mathcal{A}$ such that

(i) All operators $\pi(a)$ for $a \in C$ are compact.

(ii) The subspace $F \subset E$ generated by images of the operators $\pi(a)$ for $a \in C$ is large, i.e. $E_F = E$.

Then the action of $\mathcal{A}$ on $E$ is completely reducible.
Hints. (i) Can assume that $A$ is a $*$-subalgebra of $Op(E)$ and it is closed in norm topology.

(ii) Suppose that the algebra $A \subset Op(E)$ has an element $e$ that is an orthogonal projection on some subspace $F$. Consider the subalgebra $B = eAe \subset A$. Show that there is a natural bijection between (closed) $A$-invariant subspaces in $E_F$ and (closed) $B$-invariant subspaces in $F$. This bijection preserves the inclusion relation and orthogonality.

(iii) Show that if in (ii) the space $F$ is finite dimensional then the representation $\pi$ on the space $E_F$ is completely reducible.

(iv) Suppose the algebra $A$ contains a compact self-adjoint operator $X$. For a real number $\mu \neq 0$ consider the space $F$ of eigenvectors of $X$ with eigenvalue $\mu$. Show that this space is finite dimensional and that the orthogonal projector $e = e_F$ on this space also lies in the algebra $A$ (here we use that $A$ is norm closed in $Op(E)$).