EXC 1 IN REPRESENTATION THEORY OF FINITE AND COMPACT GROUPS

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A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign (□) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

(1) Show that the image of any character of a finite group lies inside the unit circle in $\mathbb{C}$.
(2) Let $T : (\pi, V) \to (\tau, W)$ be a morphism of representations. Suppose that it is one-to-one and onto. Show that it is an isomorphism. In other words, show that the inverse linear map T^{-1} commutes with the group action.
(3) Let X be a G-set and $F(X)$ be the space of complex-valued functions on X. Define a representation of G on $F(X)$ by $(\pi(g)f)(x) := f(g^{-1}x)$. Show that it is indeed a representation. Show that the action defined by $(\pi'(g)f)(x) = f(gx)$ does <u>not</u> define a representation for some sets X.
(4) (P) Let $(\pi_1, V_1), (\pi_2, V_2)$ be irreducible representations of a group G. Consider the direct sum (π, V) of these representations. The space V has four G-invariant coordinate subspaces $0, V_1, V_2, V$. Show that the representations π_1 and π_2 are isomorphic if and only if there exists a non-coordinate G-invariant subspace in V (i.e. a subspace distinct from the four subspaces listed above).
(5) (P) Let the group S_3 act on $\mathbb{C}^3$ by permuting the coordinates and let $\mathbb{C}^3_0$ denote the subrepresentation $\{(a, b, c) : a + b + c = 0\}$. Show that it is irreducible.
(6) (P) Show that every irreducible representation of a commutative group G is one-dimensional.
(7) (P) Define a 2-dimensional representation π of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ over a field of characteristic p by letting the generator of the group to act by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that π is not irreducible, but can not be decomposed as a direct sum of 1-dimensional representations.
(8) (□)$*$ Show that any irreducible representation of a finite commutative group over $\mathbb{R}$ has dimension 1 or 2.
(9) Let V be a representation of a finite group G over $\mathbb{C}$. Let $\langle \cdot, \cdot \rangle$ be a scalar product on V. Define
$$\langle u, v \rangle_G := \sum_{g \in G} \langle gu, gv \rangle$$
Show that $\langle \cdot, \cdot \rangle_G$ is a G-invariant scalar product and that for any subrepresentation $\tau \subset \pi$, the orthogonal complement $\tau^\perp \subset V$ is also a subrepresentation (i.e. is G-invariant).
(10) (□)$*$ Let (π, V) be an irreducible complex representation of some group G. Suppose we know that the space V has countable dimension. Show that Schur's lemma holds for π, i.e. $\dim \operatorname{Hom}(\pi, \pi) = 1$. (Hint. Prove and use the fact that any operator $A : V \to V$ has a non-empty spectrum, i.e. there exists $\lambda \in \mathbb{C}$ such that the operator $A - \lambda Id$ is not invertible).

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo.html

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