EXERCISE 3 IN REPRESENTATION THEORY

DMITRY GOUREVITCH

(1) Let V be a vector space. Define a symmetric bilinear form on $\mathrm{End}(V)$ by $\langle A, B\rangle := \mathrm{Tr}(AB)$. Show that it is non-degenerate. Show that if V is a representation of G then this form is invariant with respect to the conjugation action of G on $\mathrm{End}(V)$.

(2) Let $\rho \in Rep(G)$. Show that the natural map $\rho : \mathcal{A}(G) \to \mathrm{End}_F(\rho)$ given by $f \mapsto \sum_{g\in G} f(g)\rho(g)$ is a morphism of algebras and of representations of $G \times G$.

(3) Define a bilinear form on $\mathcal{A}(G)$ by
$$\langle f, h\rangle := \sum_{g\in G} f(g)h(g^{-1})$$
Show that this form is bilinear, symmetric and non-degenerate.

(4) (P) Let (π, V) be an irreducible complex representation of a finite group G. Show that it has an invariant Hermitian form H and that any two such forms are proportional.

(5) (P) Let G be a finite group and let (π, V) be a finite dimensional representation of G over the field of real numbers R.
   (a) Show that (π, V) is isomorphic to the dual representation (π^*, V).
   (b) Give an example of irreducible representations (π, G, V) and (τ, H, L) over the field $\mathbb{R}$ such that the tensor product representation $(\pi \otimes \tau, G \times H, V \otimes L)$ is reducible.

(6) (P) Show that if X, Y are finite G-sets and χ ia a character of G then the intertwining number $\langle \pi_X, \chi\pi_Y\rangle$ equals to the number of G-orbits O in the set $X \times Y$ such that for any point $z \in O$, the restriction $\chi|_{G_z}$ of χ to the stabilizer G_z of z is trivial.

(7) (P) Let $G = S_n$ denote the group of all permutations of a set I with n elements (symmetric group in n symbols). Consider the natural action of G on the set X of all subsets of I. It has orbits X_0, X_1, ...,X_n, where X_i consists of all subsets of size i (note that X_0 has one element). Consider the corresponding representations $\pi_i := (\pi_{X_i}, G, F(X_i))$ of the group G and decompose them into irreducible components. Describe these components for representations π_0, π_1, π_2 and compute their dimensions.

(8) A representation is called **isotypic** if it is a direct sum of isomorphic irreducible representations. Show that the following are equivalent:
   (a) π is isotypic
   (b) All irreducible subrepresentations of π are isomorphic
   (c) If $\pi \simeq \omega \oplus \tau$ with $\langle \omega, \tau\rangle = 0$ then either $\omega = 0$ or $\tau = 0$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/CompRepTheo.html

Date: November 6, 2013.