EXERCISE 4 IN REPRESENTATION THEORY

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(1) (P) Consider the representations of the group $G$ of motions of a cube on faces, edges, vertices and main diagonals of the cube, and on regular tetrahedra inscribed in the cube. Decompose them all to irreducible ones.

(2) (*) Barak has got an advanced game, where a usual game cube was replaced by an icosahedron with numbers $1, \ldots, 20$ on its faces. Each time he lost, he replaced the number on each face by the average of its neighbors. What numbers will be written on the faces after 30 losses? What is the precision of your answer? The same exercise for a dodecahedron.

Classification of irreducible representations of $S_n$. Let $X$ be a set of size $n$ and $G = \text{Sym}(X) = S_n$.

(3) Show that conjugate classes in $S_n$ = partitions of $n$, i.e. sets $(\alpha_1, \ldots, \alpha_k)$ of natural numbers s.t. $\alpha_1 + \ldots + \alpha_k = n$ and $\alpha_1 \geq \ldots \geq \alpha_k$.

Let us now find an irreducible representation for each partition $\alpha = (\alpha_1, \ldots, \alpha_k)$. Denote by $X_\alpha$ the set of all decompositions of the set $X$ to subsets $X_1, \ldots, X_k$ s.t. $|X_i| = \alpha_i$.

**Definition 1.** $T_\alpha := F(X_\alpha), \quad T'_\alpha := \text{sgn} \cdot T_\alpha$.

**Definition 2.** Denote by $\alpha^*$ the partition given by $\alpha_i^* := \{|j : \alpha_j \leq i\}$.

Let us introduce the lexicographical partition ordering on the set of partitions.

(4) Show that
(a) $\alpha^*$ is a partition and $(\alpha^*)^* = \alpha$.
(b) $^*$ is an order-reversing operation.

(5) (*)

$$\langle T_\alpha, T'_\beta \rangle = \begin{cases} 0, & \alpha > \beta^*; \\ 1, & \alpha = \beta^*. \end{cases}$$

This implies that $T_\alpha$ and $T'_\alpha$ have a unique joint irreducible component $U_\alpha$ and that these components are different for different $\alpha$. This gives a classification of all irreducible representations of $S_n$.

We will give here a formula for $\dim U_\alpha$, that we will prove later using Gelfand pairs:

$$\dim U_\alpha = \frac{n! \prod_{i<j} (l_i - l_j)}{l_1! \ldots l_k!},$$

where $l_i = \alpha_i + k - i$, $i = 1, \ldots, k$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/CompRepTheo.html

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