EXERCISE 4 IN REPRESENTATION THEORY

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(1) (P) Consider the representations of the group G of motions of a cube on faces, edges, vertices and main diagonals of the cube, and on regular tetrahedra inscribed in the cube. Decompose them all to irreducible ones.
(2) (*) Barak has got an advanced game, where a usual game cube was replaced by an icosahedron with numbers $1, ..., 20$ on its faces. Each time he lost, he replaced the number on each face by the average of its neighbors. What numbers will be written on the faces after 30 losses? What is the precision of your answer?
The same exercise for a dodecahedron.

Classification of irreducible representations of S_n. Let X be a set of size n and $G = \mathrm{Sym}(X) = S_n$.

(3) Show that conjugate classes in S_n = partitions of n, i.e. sets $(\alpha_1, ..., \alpha_k)$ of natural numbers s.t. $\alpha_1 + ... + \alpha_k = n$ and $\alpha_1 \geq ... \geq \alpha_k$.
Let us now find an irreducible representation for each partition $\alpha = (\alpha_1, ..., \alpha_k)$. Denote by X_α the set of all decompositions of the set X to subsets $X_1, .., X_k$ s.t. $|X_i| = \alpha_i$.

Definition 1. $T_\alpha := F(X_\alpha), \quad T'_\alpha := sgn \cdot T_\alpha$.

Definition 2. *Denote by α^* the partition given by $\alpha^*_i := |\{j : \alpha_j \leq i\}$.*

Let us introduce the lexicographical ordering on the set of partitions.

(4) Show that
  (a) α^* is a partition and $(\alpha^*)^* = \alpha$.
  (b) $*$ is an order-reversing operation.
(5) (*)
$$\langle T_\alpha, T'_\beta \rangle = \begin{cases} 0, & \alpha > \beta^*; \\ 1, & \alpha = \beta^*. \end{cases}$$

This implies that T_α and T'_α have a unique joint irreducible component U_α and that these components are different for different α. This gives a classification of all irreducible representations of S_n.

We will give here a formula for $\dim U_\alpha$, that we will prove later using Gelfand pairs:
$$\dim U_\alpha = \frac{n! \prod_{i<j}(l_i - l_j)}{l_1!...l_k!},$$
where $l_i = \alpha_i + k - i$, $i = 1, ..., k$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/CompRepTheo.html

Date: November 13, 2013.