EXERCISE 5 IN REPRESENTATION THEORY

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(1) (P) Find the characters of all the irreducible representations of the symmetric group $S_4$.

(2) (P) A representation is called cyclic if it is generated by some vector. Find a necessary and sufficient condition for a representation to be cyclic in terms of multiplicities of isotypic components.

(3) Let $L, V$ be linear spaces and let $X \in \text{End} V, Y \in \text{End} L$. Define $\Psi_{X,Y} : \text{Hom}(V, L) \to \text{Hom}(V, L)$ by $\Psi_{X,Y}(A) := YAX$. Show that $\text{Tr} \Psi_{X,Y} = \text{Tr} X \text{Tr} Y$.

Principal series representations of $\text{SL}(2, \mathbb{F}_q)$

(4) (P) Let $G := \text{SL}(2, \mathbb{F}_q)$ be the group of $2 \times 2$ matrices with entries in $\mathbb{F}_q$ and determinant 1. Let $X := \mathbb{F}_q^2 \setminus 0$. Consider the representation of $G$ on $F(X)$, where $F$ is some algebraically closed field of characteristic zero (e.g. $F = \mathbb{C}$). For any character $\chi$ of the multiplicative group $\mathbb{F}_q^\times$, let

$$V_{\chi} := \{ f \in F(X) \text{ s.t. } f(\lambda x) = \chi(\lambda)f(x) \}$$

Let $\Delta$ denote the standard anti-symmetric bilinear form on $F^2$. Define a linear operator $T : F(X) \to F(X)$ by

$$Tf(x) := \sum_{\{y \text{ s.t. } \Delta(x,y) = 1\}} f(y).$$

Show that

(a) $V_{\chi}$ is a subrepresentation of $F(X)$ and $F(X) = \bigoplus_{\chi} V_{\chi}$

(b) $T$ is an intertwining operator (=morphism of representations) and $T$ maps $V_{\chi}$ into $V_{\chi^{-1}}$.

(c) $\langle F(X), F(X) \rangle = 2q - 2$.

(d) $V_{\chi}$ is irreducible if $\chi \neq \chi^{-1}$ and is a sum of 2 non-isomorphic irreducible components if $\chi = \chi^{-1}$. Find the 2 components. Note that if $q$ is odd there are only two $\chi$ with $\chi = \chi^{-1}$, and if $q$ is even there is only one: $\chi = 1$.

(e) $V_{\chi} \simeq V_{\psi}$ if and only if $\chi = \psi^{\pm 1}$.

(f) $T$ is invertible

(g) The sum of squares of the dimensions of the irreducible representations we found is $q(q + 1)(q + 3)/2$, which is approximately $|G|/2$.

The $V_{\chi}$ are called principal series representations and $T$ is called the standard intertwining operator.