SUMMARY FOR THE COURSE "REPRESENTATION THEORY OF FINITE AND COMPACT GROUPS", FALL 2013

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Here I shortly summarize the the lectures that passed. Sometimes I am one lecture behind or ahead. In the latter case, the notes of the future lecture may be incomplete and may include remarks for myself, marked by "??".

1. Basic definitions and Schur's lemmas

Definition 1.1. *A **group** G is a set with a binary operation $G \times G \to G$, called multiplication, such that*

(1) $\forall f, g, h \in G. (fg)h = f(gh)$
(2) $\exists 1 \in G\, s.t. \forall g \in G,\ 1g = g1 = g$
(3) $\forall g \in G,\ \exists g^{-1} \in G\, s.t.\, gg^{-1} = g^{-1}g = 1$

*A **morphism of groups** $\phi : G \to H$ is a function $\phi : G \to H$ s.t. $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)\, \forall g_1, g_2 \in G$.*

Example 1.2. $\mathbb{Z}$ *- the group of integers,* $\mathbb{Z}/n\mathbb{Z}$ *= the cyclic group of order n, $Sym(X)$- the group of all bijections from X to itself. Also denoted by Sym_n or S_n if X has n elements. If V is a vector space of dimension n over a field F then we denote by $GL(V)$ or by $GL(n, F)$ the group of all invertible linear transformations from V to itself.*

Definition 1.3. *A G-**set** (a, X) is a set X together with a morphism of groups $a : G \to \mathrm{Sym}(X)$. We also say that G acts on X via a, and that a is an action of G on X. We will sometimes omit the a or the X from the notation. Also, we will sometimes write gx for $a(g)x$.*
*A **morphism of G-sets** $\nu : (a, X) \to (b, Y)$ is a function $\nu : X \to Y$ such that $\nu(a(g)x) = b(g)\nu(x), \forall g \in G, x \in X$.*
*Denote by X^G the **set of fixed points of** G in X, i.e. $X^G := \{x \in X : gx = x \forall g \in G\}$. For a point $x \in X$ denote by $G_x := Stab_G(x) := \{g \in G : gx = x\}$ the **stabilizer of** x in G and by $Gx := \{gx : g \in G\}$ the orbit of x.*
*An action of G on X is called **free** if all stabilizers are trivial and **transitive** if $Gx = X$ for some (and hence every) $x \in X$.*

Example 1.4.

(1) $\mathrm{Sym}(X)$ *acts on* X.
(2) $GL(V)$ *acts on* V.
(3) $G \times G$ *acts on* G *by* $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$. *This gives rise to 3 actions of G on itself, corresponding to 3 embeddings of G to $G \times G$: left, right and diagonal.*

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Definition 1.5. *Let H be a subgroup of G. Define an equivalence relation on G by $g_1 \sim g_2$ iff $g_1^{-1}g_2 \in H$. We will denote the set of equivalence classes by G/H and denote the equivalence class of g by gH. Then G/H has a natural action of G defined by $g_1(g_2H) := (g_1g_2)H$. We call it the* ***set of right H-cosets*** *in G.*
If the subgroup H is ***normal****, i.e. satisfies $ghg^{-1} \in H\, \forall g \in G, h \in H$ then G/H has a natural group structure defined by $(g_1H)(g_2H) := g_1g_2H$.*

Proposition 1.6.
1. *$|G| = |G/H| \cdot |H|$, where $| \, |$ denotes the size of a set.*
2. *Any transitive G-set X is isomorphic the set of cosets G/G_x where $x \in X$ is any point.*
3. *Any G-set is a disjoint union of transitive G-sets (its orbits).*

Many important groups have natural actions that are straightforward from their definitions. Many theorems on groups and their subgroups come from actions of G on itself or on coset spaces G/H. G-sets are important, and one can use geometry to study them. However, one cannot "compute" in G-sets. In order to compute, one needs some algebraic structure, e.g. a vector space.

Definition 1.7. *A* ***representation*** *of a group G over a field F consists of a vector space V over F and a morphism of groups $\pi : G \to GL(V)$. We will denote the representation by (G, π, V) or (π, V) or π or V. The dimension of V is called the dimension of the representation. A one-dimensional representation is called a* ***character****. A* ***morphism of*** *representations $\phi : (\pi, V) \to (\tau, W)$ is a linear map $\phi : V \to W$ that is a morphism of G-sets, i.e. such that $\phi(\pi(g)v) = \tau(g)\phi(v), \forall g \in G, v \in V$.*

Here are some examples of characters.

Example 1.8.
1. *The trivial character (of any group): $\chi(g) = 1$ for all g.*
2. *The sign character of S_n (sign of permutation).*
3. *The determinant for $GL(n, F)$.*

Here are some examples of representations.

Example 1.9.
1. *The zero representation (of any group): $V = 0$, $GL(V)$ has one element.*
2. *$SO(2, \mathbb{R})$ acts on $\mathbb{R}^2$ by rotations.*
3. *$GL(V)$ acts on V.*
4. $\mathrm{Sym}(X)$ *acts on the space $F(X)$ of all functions $X \to F$.*

Exercise 1.10. *Let $\pi, \tau \in Rep(G)$ and let $\phi : \pi \to \tau$ be a morphism of representations which is an isomorphism of linear spaces. Show that ϕ is an isomorphism of representations. In other words, show that the linear inverse ϕ^{-1} is also a morphism of representations.*

Definition 1.11. *Let (π, V) and (τ, W) be representations of G (over the same field F). Define a representation of G on the* ***direct sum*** *$V \oplus W$ by $g(v, w) := (\pi(g)v, \tau(g)w)$.*

Define a ***dual*** *or* ***contragredient*** *representation (π^*, V^*) by $(\pi^*(g)\phi)(v) := \phi(g^{-1})v$.*

Let (σ, U) be a representation of H (over F). Define a representation of $G \times H$ on the ***tensor product*** *$V \otimes U$ by $(g, h)(v \otimes u) := \pi(g)v \otimes \sigma(g)u$.*

In particular, if $G = H$ then $\pi \otimes \sigma$ is a representation of $G \times G$, which also becomes a representation of G using the diagonal embedding $\Delta : G \hookrightarrow G \times G$. This enables us to define an action of G on $\mathrm{Hom}_F(V, U) = V^* \otimes U$.

Exercise 1.12. *Check that* $\mathrm{Hom}_F(V, U)^G = \mathrm{Hom}_G(\pi, \sigma)$.

Definition 1.13. *A* ***subrepresentation*** *of* (G, π, V) *is a* G*-invariant subspace of* V*, with induced action of* G.

Example 1.14. *Any representation has (at least) 2 subrepresentations :* 0 *and* V.

Definition 1.15. *A representation is called* ***irreducible*** *if it has only 2 subrepresentations.*

Example 1.16.

(1) *Any character is irreducible*
(2) *The action of* $SO(2, \mathbb{R})$ *on* $\mathbb{R}^2$ *by rotations is irreducible, while the action of* $\mathbb{R}^\times$ *on* $\mathbb{R}^2$ *by homotheties is not.*

Exercise 1.17. *Every irreducible representation of a finite group is finite dimensional.*

In the next lecture we will show that every representation is a direct sum of irreducible ones, and for a given group there is a finite number of isomorphism classes of irreps (unlike prime numbers). Thus, the main goals of representation theory are to classify all irreducible representations of a given group (up to isomorphism) and given a representation to find its decomposition to irreducible ones.

The most important properties of irreducible representations are Schur's lemmas.

Lemma 1.18. *Let* ρ *and* σ *be irreps of a group* G.

(1) *Any non-zero morphism* $\phi : \rho \to \sigma$ *is an isomorphism.*
(2) *If the field* F *is algebraically closed and* ρ *is finite-dimensional then* $\mathrm{Hom}(\rho, \rho) = F \cdot Id$.

Proof. (1) $\mathrm{Ker}\, \phi$ is a subrepresentation of ρ and $\mathrm{Im}\, \phi$ is a subrepresentation of σ.
(2) Let $\varphi \in \mathrm{Hom}(\rho, \rho)$ and λ be an eigenvalue of φ. Since $\varphi - \lambda Id$ is not invertible, (1) implies that it is zero. □

Corollary 1.19. *Every irrep of a finite commutative group over an algebraically closed field is one-dimensional.*

Exercise 1.20. *Every irrep of a commutative group over* $\mathbb{R}$ *is at most 2-dimensional.*

Exercise 1.21. *Let* $(\pi_1, V_1), (\pi_2, V_2)$ *be irreps of a group* G*. Consider the direct sum* (π, V) *of these representations. The space* V *has four* G*-invariant coordinate subspaces* $0, V_1, V_2, V$*. Show that the representations* π_1 *and* π_1 *are isomorphic if and only if there exists a non-coordinate* G*-invariant subspace in* V *(i.e. a subspace distinct from the four subspaces listed above).*

In this course we will consider mainly representations over the field $\mathbb{C}$ of complex numbers.

2. Existence and uniqueness of decomposition to irreducibles, intertwining numbers and the group algebra.

From now on we consider only finite groups.

Definition 2.1 (Exercise)**.** *A representation π is called completely reducible if one of the following equivalent conditions holds.*

(1) π is a direct sum of irreducible representations.
(2) For every subrepresentation $\tau \subset \pi$ there exists another subrepresentation $\tau' \subset \pi$ such that $\pi = \tau \oplus \tau'$.

Note that an irreducible representation is completely reducible :-).

Theorem 2.2 (Weyl-Mashke)**.** *Every representation (π, V) of a finite group G is completely reducible.*

Proof. Let $\tau \subset \pi$. It is enough to find a G-invariant linear projection on τ. We take any linear projection on τ and average it. Namely, we take a linear map $p : V \to V$ s.t. $p^2 = p$ and $\operatorname{Im} p = \tau$ and replace it by $p' := \sum_{g \in G} \pi(g) p \pi(g^{-1})$. Check that $p'^2 = p'$, $\operatorname{Im} p' = \tau$ and p' is G-invariant. □

The idea of averaging is very important. It always gives something G-invariant, but sometimes produces zero. It already takes advantage of linearity of our subject - we would not be able to do such a thing with G-sets.

In fact, this works over any field F such that $|G|$ is not zero in F. The assumptions that G is finite and $|G|$ is not zero in F are necessary, as shown by the following example.

Example 2.3. *Define $A \in Mat_2(F)$ by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let the group $\mathbb{Z}$ act on F^2 by $\pi(n) := A^n$. Then this representation is not completely reducible.*
If $char F = p$ then the same example gives a representation of the finite group $\mathbb{Z}/p\mathbb{Z}$.

Corollary 2.4. *Any matrix A with $A^n = Id$ is diagonalizable.*

In order to prove uniqueness of the decomposition we introduce a very important notion, called intertwining number.

Notation 2.5. *We denote by $Rep(G)$ the collection of all representations of G and by $Irr(G)$ the set of isomorphism classes of irreducible representations of G.*

Definition 2.6. *Let $\pi, \tau \in Rep(G)$. Define the* ***intertwining number*** *of π and τ by $\langle \pi, \tau \rangle := \dim \operatorname{Hom}_G(\pi, \tau)$.*

Lemma 2.7. *The "form" $\langle \cdot, \cdot \rangle$ is "bilinear and symmetric". Namely*

(1) $\langle \pi_1 \oplus \pi_2, \tau \rangle = \langle \pi_1, \tau \rangle + \langle \pi_2, \tau \rangle$
(2) $\langle \pi, \tau_1 \oplus \tau_2 \rangle = \langle \pi, \tau_1 \rangle + \langle \pi, \tau_2 \rangle$
(3) $\langle \bigoplus a_i \pi_i, \bigoplus b_i \tau_i \rangle = \sum a_i b_i \langle \pi_i, \tau_i \rangle$, where a_i and b_i are natural numbers or zeros.
(4) $\langle \pi, \tau \rangle = \langle \tau, \pi \rangle$

Proof. (1)-(2) are obvious and apply (3), which in turn implies (4) using complete reducibility and Schur's lemmas. □

Note that we just proved that the spaces $\mathrm{Hom}_G(\pi,\tau)$ and $\mathrm{Hom}_G(\tau,\pi)$ are equidimensional and hence isomorphic, but we have no natural isomorphism between them.

Corollary 2.8. *The decomposition of any representation to a direct sum of irreducible ones is unique. The multiplicity with which an irrep σ appears in a representation π equals $\langle\sigma,\pi\rangle$.*

Corollary 2.9. *A representation π is irreducible if and only if $\langle\pi,\pi\rangle = 1$.*

For a vector space V denote $\mathrm{End}(V) := \mathrm{Hom}(V,V)$. Note that $\mathrm{End}(V) = V \otimes V^*$. Thus, let us study some properties of actions on tensor products.

Let $\pi \in Rep(G)$ and $\tau \in Rep(H)$.

Exercise 2.10. *Show that $(\pi\otimes\tau)|_G = (\dim\tau)\pi$ and $(\pi\otimes\tau)|_H = (\dim\pi)\tau$.*

Exercise 2.11. *Show that $(\pi\otimes\tau)^{G\times H} = \pi^G\otimes\tau^H$.*

Lemma 2.12. *Let $\rho\in Irr(G)$ and $\sigma\in Irr(H)$. Then $\rho\otimes\sigma\in Irr(G\times H)$.*

Proof.

$$\mathrm{End}_{G\times H}(\rho\otimes\sigma) = (\mathrm{End}_F(\rho\otimes\sigma))^{G\times H} = (\rho^*\otimes\sigma\otimes\sigma^*\otimes\rho)^{G\times H} = (\rho^*\otimes\rho\otimes\sigma\otimes\sigma^*)^{G\times H} =$$
$$(\rho^*\otimes\rho)^G\otimes(\sigma\otimes\sigma^*)^H = \mathrm{End}_F(\rho)^G\otimes\mathrm{End}_F(\sigma)^H = \mathrm{End}_G(\rho)\otimes\mathrm{End}_H(\sigma).$$

Thus, $\langle\rho\otimes\sigma,\rho\otimes\sigma\rangle = \langle\rho,\rho\rangle\langle\sigma,\sigma\rangle = 1$. □

Exercise 2.13. *Prove that every irrep of $G\times H$ can be obtained in this way.*

Corollary 2.14. *If $\rho\in Irr(G)$ then $\mathrm{End}_F(\rho)\in Irr(G\times G)$.*

Definition 2.15 (Group algerba)**.** *Define the **group algebra** $\mathcal{A}(G)$ of G to be the algebra spanned over F by the symbols δ_g, $g\in G$ with multiplication defined by $\delta_g\delta_h = \delta_{gh}$. Note that this is an associative non-commutative (unless G is commutative) algebra with unit (equal to δ_1). We can also view it as the algebra of functions from G to F, or the algebra of measures on G, with multiplication given by convolution:*

$$f*h(g) := \sum_{x\in G} f(gx^{-1})h(x)$$

*We define a representation of $G\times G$ on $\mathcal{A}(G)$ by $(g_1,g_2)\delta_x := \delta_{g_1xg_2^{-1}}\,\forall x\in G$ or, equivalently, $((g_1,g_2)f)(x) := f(g_1^{-1}xg_2)\,\forall f\in\mathcal{A}(G)$, $x\in G$. This representation is called the **regular representation** of G. Its restrictions on first and second coordinate of $G\times G$ are called the **left regular** and **right regular** representations respectively.*

Definition 2.16. *A representation of an algebra with unit A on a vector space V is a morphism of algebras with unit $A\to\mathrm{End}(V)$.*

Exercise 2.17. *A representation (π,V) of G defines a representation of $\mathcal{A}(G)$ on V and vice versa.*

Lemma 2.18. *If $\rho\in Irr(G)$ then the natural morphism of algebras $\mathcal{A}(G)\to\mathrm{End}_F(\rho)$ is onto.*

Proof. $\mathrm{End}_F(\rho)$ is an irrep of $G\times G$ and the image of this morphism is a non-zero subrepresentation. □

3. Decomposition of the regular representation. Corollaries on number and dimensions of irreducible representations. Examples for small symmetric groups

Lemma 3.1. *Let V be a vector space. Then $\langle A, B\rangle := \mathrm{Tr}(AB)$ defines a non-degenerate symmetric bilinear form on $\mathrm{End}(V)$. Moreover, if V is a representation of G then this form is invariant with respect to the diagonal action of G. This form is called the **trace form**.*

Theorem 3.2. *The natural morphism*

$$\phi : \mathcal{A}(G) \to \bigoplus_{\sigma \in Irr(G)} \mathrm{End}_F(\sigma)$$

is an isomorphism of algebras and of representations of $G \times G$.

Proof.
(1) It is easy to see that ϕ is a morphism of algebras and of representations of $G \times G$. Thus it is enough to show that ϕ is one to one and onto.
(2) Suppose $f \in \mathrm{Ker}\,\phi \subset \mathcal{A}(G)$. Then f acts by zero on any irreducible representation of G and thus on any representation of G. Thus, f acts by zero on $\mathcal{A}(G)$, but $f\delta_1 = f$ and thus $f = 0$.
(3) Define a morphism $\psi : \bigoplus_{\sigma\in Irr(G)} \mathrm{End}_F(\sigma) \to \mathcal{A}(G)$ in the following way. For $A \in \mathrm{End}(\sigma)$ let by $\psi(A)(g) := \mathrm{Tr}(\sigma(g)A)$, and continue by linearity to the direct sum. Let us show that it is an embedding.
From Lemmas 2.18 and 3.1 we see that $\mathrm{Ker}\,\psi$ does not intersect any coordinate of the direct sum. On the other hand, by Exercise 1.21, $\mathrm{Ker}\,\psi$ must be a coordinate subspace. Thus $\mathrm{Ker}\,\psi = 0$.
(4) Now, by (3) the R.H.S. is finite dimensional and its dimension is at most the dimension of L.H.S, and by (2), ϕ is one to one. Thus ϕ is an isomorphism.

□

Corollary 3.3. *(1) $Irr(G)$ is finite and*

$$\sum_{\sigma_{Irr(G)}} (\dim\sigma)^2 = |G|.$$

(2) $|Irr(G)|$ equals the number of conjugacy classes in G.

Proof. (1): obvious. (2): both are equal to the dimension of the center of $\mathcal{A}(G)$. □

Example 3.4. *If G is commutative then $|Irr(G)| = |G|$ and all irreps are characters.*

Lemma 3.5. *Let X and Y be G-sets. Then $\langle F(X), F(Y)\rangle$ equals the number of orbits of G in $X \times Y$ under the diagonal action.*

Corollary 3.6. *If the action of G on X is double-transitive then $F_0(X)$ is irreducible.*

Example 3.7. *Classification of $Irr(S_2), Irr(S_3), Irr(S_4)$.*

4. Isotypic components; Characters, Schur orthogonality relations

4.1. Isotypic components.

Definition 4.1. *A representation is called* ***isotypic*** *if it is a direct sum of isomorphic irreducible representations.*

Exercise 4.2. *The following are equivalent:*
(1) π is isotypic
(2) All irreducible subrepresentations of π are isomorphic
(3) If $\pi \simeq \omega \oplus \tau$ with $\langle \omega, \tau \rangle = 0$ then either $\omega = 0$ or $\tau = 0$.

Theorem 4.3. *Let $(\pi, V) \in Rep(G)$. Then there exists a unique set of subrepresentations V_i such that $V = \bigoplus_{i=1}^{k} V_i$, V_i are isotypic, and $\langle V_i, V_j \rangle = 0$. Moreover, for any subrepresentation $W \subset V$, we have $W = \bigoplus_{i=1}^{k} (W \cap V_i)$.*

Proof. By induction. Existence is easy. Uniqueness follows from the "moreover" part. To prove the "moreover" part, fix a decomposition $V = \bigoplus V_i$, let $W \subset V$ and consider the decomposition $W = \bigoplus W_i$ where W_i has the same type as V_i, or is zero. Then $W \cap V_i \subset W_i$. On the other hand, W_i has zero projection on V_j, for $j \neq i$ and thus $W_i \subset V_i$. Thus $W_i = V_i$. □

The V_i are called the isotypic components of π.

Definition 4.4. *If all isotypic components of π are irreducible then π is called multiplicity free.*

Lemma 4.5 (Easy). *Every intertwining operator $L \in \mathrm{Hom}_G(\pi, \pi)$ preserves each isotypic component. In particular, if π is multiplicity free then L is scalar on each V_i.*

Exercise 4.6. *Barak has got a game for his birthday. In the game there was a cube with digits 1,...,6 on its faces. Each time he played with his friends and lost, he blamed the cube and modified it by replacing the number on every face by the average of its 4 neighbors. What numbers will be written on the faces after 10 losses?*

Solution. Let V denote the 6-dimensional space of functions on the set X of faces of the cube and L denote the "averaging on neighbors" operator. Of course, we can guess that the answer will be approximately the constant function 3.5. However, to know how precise this approximation is we will need to diagonalize L and representation theory will help us.

Let G denote the group of motions of the cube and consider V as its representation. Then G has 3 orbits on X, thus $\langle V, V \rangle = 3$ and thus V is a sum of 3 non-isomorphic irreducible representations. One is, of course, the 1-dimensional space V_1 of constant functions. The other is the 2-dimensional space V_2 of "symmetric" functions with zero sum, namely functions that have the same value on opposite faces (and zero sum). The third is the 3-dimensional space V_3 of "anti-symmetric" functions.

The operator L commutes with the group action and thus acts by a scalar λ_i on each V_i. Taking convenient vectors from each V_i we get $\lambda_1 = 1, \lambda_2 = 1/2, \lambda_1 = 0$. Note that V has the natural form $\langle f, g \rangle := \sum f(x)\overline{g(x)}$, which is G-invariant and thus can be used to compute projections to V_i. Let ξ be the original function given by $(1, 2, 3, 4, 5, 6)$. Then its projection ξ_1 to V_1 is the constant function 3.5. The length of the projection to V_2 is at most $\sqrt{2((3.5-1)^2 + (3.5-2)^2 + (3.5-3)^2)} = \sqrt{17.5}$ and thus $|L^{10}(\xi) - \xi_1| \leq \sqrt{17.5}/2^{10} < 0{,}005$. □

Exercise 4.7. *Classify all irreducible representations of the group G from the solution of the last exercise.*

Hint. Use the action of G on faces, edges, vertices and main diagonals of the cube, and on regular tetrahedra inscribed in the cube.

Exercise 4.8. *Repeat Exercise 4.6 for an icosahedron and a dodecahedron.*

4.2. Characters.

Definition 4.9. *Let $(\pi, V) \in Rep(G)$. Define a function χ_π on G by $\chi_\pi(g) := \operatorname{Tr}\pi(g)$.*

Lemma 4.10.

(1) *If $\pi \simeq \tau$ then $\chi_\pi = \chi_\tau$.*
(2) *$\chi_\pi(hgh^{-1}) = \chi_\pi(g)$, i.e. $\chi_\pi \in Z(\mathcal{A}(G))$.*
(3) *$\chi_{\pi\oplus\tau} = \chi_\pi + \chi_\tau$.*
(4) *$\chi_{\pi\otimes\tau} = \chi_\pi\chi_\tau$.*
(5) *$\chi_\pi(g^{-1}) = \chi_{\pi^*}(g)$.*

This lemma immediately follows from the corresponding properties of trace.

Definition 4.11. *Define a bilinear form on $\mathcal{A}(G)$ by*

$$\langle f, h\rangle := \sum_{g\in G} f(g)h(g^{-1})$$

Exercise 4.12. *This form is bilinear, symmetric and non-degenerate.*

4.3. Schur orthogonality relations.

Theorem 4.13 (Schur orthogonality relations)**.**

$$\langle\chi_\pi, \chi_\tau\rangle = \langle\pi, \tau\rangle$$

Proof. Let us first prove for the case when π is the trivial representation. Then $\operatorname{Hom}(\pi, \tau) = \tau^G$. Define $p : \tau \to \tau^G$ by $p := 1/|G| \sum \tau(g)$. Then $\operatorname{Im} p = \tau^G$ and $p|_{\tau^G} = Id$, i.e. p is a projection on τ^G. Thus, $\dim \tau^G = \operatorname{Tr}(p)$. On the other hand,

$$\operatorname{Tr}(p) = 1/|G| \sum \operatorname{Tr}(\tau(g)) = 1/|G| \sum_{g\in G} \chi_\tau(g) = 1/|G| \sum \chi_\pi(g^{-1})\chi_\tau(g) = \langle\chi_\pi, \chi_\tau\rangle$$

Now we will repeat the same argument for the general case, using the following exercise.
Exercise Let L, V be linear spaces and let $X \in \operatorname{End} V, Y \in \operatorname{End} L$. Define $\Psi_{X,Y}$: $\operatorname{Hom}(L, V) \to \operatorname{Hom}(V, L)$ by $\Psi_{X,Y}(A) := YAX$. Then $\operatorname{Tr}\Psi_{X,Y} = \operatorname{Tr} X \operatorname{Tr} Y$.
Hint There are (at least) to ways to solve this:
1) There is a "free' proof with tensor calculus.
2) In coordinates, $(YE_{ij}X)_{ij} = Y_{ii}X_{jj}$.

Now, let V be the space of π and L be the space of τ. Then $\operatorname{Hom}_G(\pi, \tau) = \operatorname{Hom}(V, L)^G$. For any $g \in G$ define $Q(g) : \operatorname{Hom}(V, L) \to \operatorname{Hom}(V, L)$ by $Q(g)(A) := \tau(g)A\pi(g^{-1})$. Then $1/|G| \sum_{g\in G} Q(g)$ is a projector from $\operatorname{Hom}(V, L)$ onto $\operatorname{Hom}_G(\pi, \tau) = \operatorname{Hom}(V, L)^G$. Thus

$$\langle\pi, \tau\rangle = \dim \operatorname{Hom}_G(\pi, \tau) = \operatorname{Tr}(1/|G| \sum_{g\in G} Q(g)) = 1/|G| \sum_{g\in G} \chi_{tau}(g)\chi_\pi(g^{-1}) = \langle\chi_\pi, \chi_\tau\rangle$$

∎

Corollary 4.14. *The character is a full invariant of a representation.*

Proof. $\pi = \bigoplus_{\rho \in IrrG} m_\rho \rho$, and m_ρ are determined by $m_\rho = \langle \pi, \rho \rangle = \langle \chi_\pi, \chi_\rho \rangle$. □

Corollary 4.15. *Characters of irreducible representations form an orthonormal basis for $Z(\mathcal{A}(G))$.*

Proof. By Lemma 4.10, characters of irreducible representations belong to $Z(\mathcal{A}(G))$. By the theorem and Schur's lemmas, they form an orthonormal set. By Corollary 3.3 their number is equal to $\dim Z(\mathcal{A}(G))$. Thus, they form an orthonormal basis. □

5. More Character theory; Classification of representations of symmetric groups

5.1. More character theory.

Lemma 5.1. *If $F = \mathbb{C}$ then $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$. Thus, on $Z(\mathcal{A}(G))$ the form $\langle\, , \rangle$ coincides with the scalar product defined by $\langle f, h \rangle' = \sum_{g \in G} f(g)\overline{h(g)}$.*

Proof. As we showed some time ago, π has an invariant scalar product and thus $\pi^* \simeq \overline{\pi}$. Now, $\chi_\pi(g^{-1}) = \chi_{\pi^*}(g) = \chi_{\overline{\pi}}(g) = \overline{\chi_\pi(g)}$. □

Proposition 5.2. *Let $\rho \in Irr(G)$ and let $z_\rho = \dim \rho / |G| \sum_{g \in G} \chi_\rho(g^{-1}) \delta_g$. Then $\rho(z_\rho) = Id$ and $\sigma(z_\rho) = 0$ for any $\sigma \not\simeq \rho \in Irr(G)$.*

Proof. Let $\omega \in Irr(G)$. Then, by the second Schur's lemma, $\omega(z_\rho)$ is a scalar. Now, $\operatorname{Tr} \omega(z_\rho) = \dim \rho / |G| \sum_{g \in G} \chi_\rho(g^{-1}) \chi_\omega(g) = \dim \rho \cdot \langle \rho, \omega \rangle$. Thus, $\omega(z_\rho) = Id$ if $\rho \simeq \omega$ and $\omega(z_\rho) = 0$ otherwise. □

Corollary 5.3. *The inverse of the map $\mathcal{A}(G) \simeq \bigoplus_{\rho \in Irr(G)} \operatorname{End}_F(\rho)$ is given on the coordinate $\operatorname{End}_F(\rho)$ by $A \mapsto f_A(g) = \dim \rho / |G| \operatorname{Tr}(A \rho(g^{-1}))$.*

Corollary 5.4. $\forall \rho \in Irr(G)$, $\dim \rho$ *divides* $|G|$.

For the proof we will need

Definition 5.5. *A **lattice** is an abelian group without torsion.*

Theorem 5.6 (from commutative algebra)**.** *Any finitely generated lattice L has a basis, i.e. $L \simeq \mathbb{Z}^n$. In other words, $\exists l_1, ..., l_n \in L$ s.t. $\forall l \in L, l = \sum a_i l_i$, $l_i \in \mathbb{Z}$.*

Lemma 5.7. *Let V be a vector space, and $L < V$ a finitely generated lattice. Let $A : V \to V$ s.t. $A(L) \subset L$. Suppose that $A^2 = qA$. Then $q \in \mathbb{Z}$.*

Proof. Fix a basis $(l_1, ..., l_n)$ for L. Take $x \in L$ and let $y := Ax$. Then $Ay = qy$ and $A^k y = q^k y \, \forall k \geq 1$. Thus q is rational, and any power of the denominator of q divides all the coordinates of y. Thus $q \in \mathbb{Z}$. □

Proof of Corollary 5.4. $V := \mathcal{A}(G)$, $A :=$ convolution with $\sum \chi_\rho(g^{-1})$, $q = |G| / \dim \rho$ and $L :=$ lattice generated by $\{\xi \delta_g : \xi$ is a root of unity of order $|G|\}$. □

5.2. **Classification of irreducible representations of S_n.** Let X be a set of size n and $G = \mathrm{Sym}(X) = S_n$.

Lemma 5.8. *Conjugate classes in S_n = partitions of n, i.e. sets $(\alpha_1, ..., \alpha_k)$ of natural numbers s.t. $\alpha_1 + ... + \alpha_k = n$ and $\alpha_1 \geq ... \geq \alpha_k$.*

Let us now find an irreducible representation for each partition $\alpha = (\alpha_1, ..., \alpha_k)$. Denote by X_α the set of all decompositions of the set X to subsets $X_1, .., X_k$ s.t. $|X_i| = \alpha_i$.

Definition 5.9. $T_\alpha := F(X_\alpha), \quad T'_\alpha := sgn \cdot T_\alpha.$

Definition 5.10. *Denote by α^* the partition given by $\alpha^*_i := |\{j : \alpha_j \leq i\}$.*

Exercise 5.11. *α^* is a partition and $(\alpha^*)^* = \alpha$.*

Let us introduce the lexicographical ordering on the set of partitions. Note that $*$ is an order-reversing operation.

Theorem 5.12.

$$\langle T_\alpha, T'_\beta \rangle = \begin{cases} 0, & \alpha > \beta^*; \\ 1, & \alpha = \beta^*. \end{cases}$$

We leave the proof as a difficult exercise.

The theorem implies that T_α and T'_α have a unique joint irreducible component U_α and that these components are different for different α. This gives a classification of all irreducible representations of S_n. This classification is not very satisfying, but a long and detailed study of the intertwining operator of T_α and T'_α will lead to a (quite long) expression for the character of U_α. We will give here a formula for $\dim U_\alpha$, that we will prove later using Gelfand pairs:

$$\dim U_\alpha = \frac{n! \prod_{i<j}(l_i - l_j)}{l_1!...l_k!},$$

where $l_i = \alpha_i + k - i$, $i = 1, ..., k$.

6. Commutative groups: Fourier transform. Induction

In this lecture we had two completely different topics.

6.1. **Commutative groups: Fourier transform.** Let G be a finite commutative group. Then, by the second Schur's lemma all irreducible representations are 1-dimensional (characters). Their number is equal to $|G|$. Actually, the characters form a group: $(\chi \cdot \psi)(g) := \chi(g)\psi(g)$. It is called the (Pontryagin) dual group $\widehat{G}$. This group is not canonically isomorphic to G, but $G \approxeq \widehat{\widehat{G}}$ canonically.

Now, we constructed an isomorphism $\mathcal{A}(G) \approxeq \bigoplus \mathrm{End}(\sigma)$. For commutative G it becomes $\mathcal{F} : \mathcal{A}(G) \approxeq \mathcal{A}(\widehat{G})$. It is called Fourier transform. To see why let us write the explicit formula.

$$\mathcal{F}(f)(\chi) = \sum_{g \in G} f(g)\chi(g)$$

By Schur orthogonality relations, we know that the characters form an orthonormal basis for $\mathcal{A}(G)$ and thus f can be reconstructed from $\mathcal{F}(f)$ by

$$f(g) = \sum_{\chi\in\widehat{G}} \mathcal{F}(f)(\chi)\chi(g)^{-1}$$

since $\mathcal{F}(f)(\chi)$ is exactly the χ^{-1}-coordinate of f. This formula is called Fourier inversion formula. It also shows that $\mathcal{F}(\mathcal{F}(f))(g) = f(g^{-1})$, under the identification $G \approxeq \widehat{\widehat{G}}$.

To make things more familiar, let take $F = \mathbb{C}$. Then we have $\chi^{-1} = \overline{\chi}$. Let us consider $G = \mathbb{Z}/n\mathbb{Z}$ and choose a non-trivial character ψ by $\psi(k) := exp(\frac{2\pi ik}{n})$. Then for $c \in \mathbb{Z}/n\mathbb{Z}$ we have another character is given by $a \mapsto \psi(ca)$, and all characters of G are of this form. This gives an identification of G with $\widehat{G}$ and the familiar formulas for Fourier transform. The same thing happens for $G = \mathbb{R}$, but analysis comes in. For $G = S^1$, $\widehat{G} = \mathbb{Z}$ and Fourier transform becomes Fourier series.

Application. Multiplication of numbers.

Remark. The isomorphism $\mathcal{A}(G) \approxeq \bigoplus \mathrm{End}(\sigma)$ for non-commutative groups can be viewed as a generalization of Fourier transform.

6.2. **Induced representation.** We are looking for a way of "lifting" representations of a subgroup $H < G$ to representations of G. In other words, we are looking for a "functor" $Ind_H^G : Rep(H) \to Rep(G)$.

Let us first find the trace (character) ψ of $\mathrm{Ind}_H^G(\pi)$. We have a natural map Res_H^G : $Z(\mathcal{A}(G)) \to Z(\mathcal{A}(H))$. On both algebras we have a natural non-degenerate bilinear form. Let us define $\mathrm{Ind}_H^G : Z(\mathcal{A}(H)) \to Z(\mathcal{A}(G))$ as the conjugate to Res_H^G w.r. to these forms. For any $g \in G$ let C_g denote the conjugacy class of g and δ_{C_g} denote the function which equals $|C_g|^{-1}$ on C_g and zero outside C_g. Then the functions of this form span $Z(\mathcal{A}(G))$. Now, by definition

$$\psi(g) = |G|\langle C_g\, , \psi^{-1}\rangle_G = |G|\langle C_g|_H, \chi_{\pi^*}\rangle = \frac{|G|}{|H||C_g|}\sum_{h\in C_g\cap H} \chi_\pi(h)$$

As we know, this defines $\mathrm{Ind}_H^G(\pi)$ uniquely (up to isomorphism). One only has to show existence now. However, before doing this let us check the meaning of induction by evaluating $\mathrm{Ind}_H^G(\chi_\pi)$ on another (generating) subset of $Z(\mathcal{A}(G))$ - the one formed by characters of representations.

$$\langle \tau, \mathrm{Ind}_H^G(\pi)\rangle = \langle \chi_\tau, \mathrm{Ind}_H^G(\chi_\pi)\rangle_G = \langle Res_H^G\chi_\tau, \chi_\pi\rangle_H = \langle Res_H^G\tau, (\pi)\rangle$$

This very important formula is called Frobenius reciprocity. First of all, it shows that $\mathrm{Ind}_H^G(\chi_\pi)$ is the character of a representation. It also defines induction uniquely and in fact could be guessed without considering characters since in means that $\mathrm{Ind}_H^G(\pi)$ is the "free representation of G generated by π". Similar definitions work for the free group, free module etc.

Let us now construct $\mathrm{Ind}_H^G(\pi)$. First let us consider several examples

Example 6.1. *(1)* $H = \{e\}$, $\mathrm{Ind}_H^G(\mathbb{C}) = F(G)$.
(2) For any $H, \mathrm{Ind}_H^G(\mathbb{C}) = F(G/H)$.
(3) For any character χ *of* H, $\mathrm{Ind}_H^G(\chi) = \{f \in F(G) : f(gh) = \chi(h^{-1})f(g)$.

(4) *For any H-set X, the free G-set generated by X is the set of H-orbits in $G \times X$ under the action $h(g, x) := (gh^{-1}, hx)$.*

Based on these we define, for any $(\pi, V) \in Rep(H)$,

$$\operatorname{Ind}_H^G(\pi) = \{f \in F(G, V) : f(gh) = \pi(h^{-1})f(g)\},$$

where $F(G, V)$ denotes all the functions from G to V with the usual action of G, i.e. $\operatorname{Ind}_H^G(\pi)(g)f(g') = f(g^{-1}g')$.

Moreover, this construction is functorial. This means that for $\pi_1, \pi_2 \in Rep(H)$ and $\phi \in \operatorname{Hom}_H(\pi_1, \pi_2)$ we define $\operatorname{Ind}_H^G(\phi) : \operatorname{Ind}_H^G(\pi_1) \to \operatorname{Ind}_H^G(\pi_2)$ by $\operatorname{Ind}_H^G(\phi)(f)(g) = \phi(f(g))$, and this preserves composition.

Lemma 6.2. *The above construction satisfies Frobenius reciprocity. More precisely, for any $\pi \in Rep(H)$ and $\tau \in Rep(G)$ there is a canonical isomorphism $\operatorname{Hom}_G(\tau, Ind_H^G(\pi)) \simeq \operatorname{Hom}_H(\tau|_H, \pi)$.*

Proof. To build the isomorphism let $\phi : \tau \to Ind_H^G(\pi)$. Then its image is given by $\psi(w) = (\phi(w))(e)$, where $e \in G$ is the identity element. The inverse morphism maps $\psi \in \operatorname{Hom}_H(\tau|_H, \pi)$ to $\phi \in \operatorname{Hom}_G(\tau, Ind_H^G(\pi))$ defined by $\phi(w)(g) := \psi(g^{-1}w)$. ☐

Exercise 6.3. (1) *For $H < G$ and $\pi_1, \pi_2 \in Rep(H)$,*

$$\operatorname{Ind}_H^G(\pi_1 \oplus \pi_2) = \operatorname{Ind}_H^G(\pi_1) \oplus \operatorname{Ind}_H^G(\pi_2).$$

(2) *For $H_1 < H_2 < G$ and $\pi \in Rep(H)$,*

$$\operatorname{Ind}_{H_2}^G \operatorname{Ind}_{H_1}^{H_2} \pi = \operatorname{Ind}_{H_1}^G \pi$$

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URL: http://www.wisdom.weizmann.ac.il/~dimagur/CompRepTheo.html