Vanishing and Eulerianity of Fourier coefficients of automorphic forms.

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Representation theory seminar, BGU, June 2020
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Definitions

- $\mathbb{K}$: number field, $\mathbb{A} := \mathbb{A}_\mathbb{K}$, $\mathbf{G}$: reductive group over $\mathbb{K}$, $\Gamma := \mathbf{G}(\mathbb{K})$, $G := \mathbf{G}(\mathbb{A})$, $\mathfrak{g} := \text{Lie}(\Gamma)$.
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• Define $\mathfrak{n} := \mathfrak{n}_{H,f} := (\mathfrak{g}_1 \cap \mathfrak{g}_f^f) \oplus \bigoplus_{i > 1} \mathfrak{g}_i$, $\mathcal{N} := \text{Exp}(\mathfrak{n})(\mathbb{A})$. 

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- Fix a non-trivial unitary additive character $\psi : \mathbb{K} \backslash \mathcal{A} \to \mathbb{C}$ and define $\chi_f : N \to \mathbb{C}$ by $\chi_f(\text{Exp} X) := \psi(\langle f, X \rangle)$. 

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Fourier coefficients

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Definitions

- $K$: number field, $A := A_K$, $G$: reductive group over $K$, $\Gamma := G(K)$, $G := G(A)$, $\mathfrak{g} := \text{Lie}(\Gamma)$.
- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i := \mathfrak{g}_i^H$ denote the eigenspaces of $\text{ad}(H)$. Assume that all the eigenvalues $i$ lie in $\mathbb{Q}$.
- Let $f \in \mathfrak{g}_{-2}$. Call $(H, f) \in \mathfrak{g} \times \mathfrak{g}$ a Whittaker pair.
- Define $n := n_{H,f} := (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i > 1} \mathfrak{g}_i$, $N := \text{Exp}(n)(A)$.
- Fix a non-trivial unitary additive character $\psi : K \setminus A \to \mathbb{C}$ and define $\chi_f : N \to \mathbb{C}$ by $\chi_f(\text{Exp} X) := \psi(\langle f, X \rangle)$.
- Let $[N] := (\Gamma \cap N) \setminus N$. For automorphic form $\eta$ on $G$, define Fourier coefficient
  \[
  \mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1}dn.
  \]
Two central cases of Fourier coefficients

\[ [H, f] = -2f, \ n = (g_1 \cap g^f) \oplus \bigoplus_{i>1} g_i, \ N = \text{Exp}(n)(A), \]

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- Neutral Fourier coefficient, coming from \(\mathfrak{sl}_2\)-triple \((e,H,f)\), e.g.:

\[H = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\quad f = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad n = \begin{pmatrix}
0 & * & 0 & * \\
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  \end{pmatrix}
  \]

- Whittaker coefficient \( \mathcal{W}_f \), with \( N \) maximal unipotent, e.g.:

  \[
  H = \begin{pmatrix}
  3 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & -1 & 0 & 0 \\
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  \end{pmatrix},
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Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.
Two central cases of Fourier coefficients

\[ [H, f] = -2f, \quad n = (g_1 \cap g^f) \oplus \bigoplus_{i>1} g_i, \quad N = \text{Exp}(n)(A), \]

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Examples of Fourier coefficients

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Comparison for \( G = \text{GL}_3(\mathcal{A}) \):

- Neutral Fourier coefficient:

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Examples of Fourier coefficients

\[ [H, f] = -2f, \ n = (g_1 \cap g^f) \oplus \bigoplus_{i>1} g_i, \ N = \text{Exp}(n)(A) \]

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Comparison for \( G = \text{GL}_3(A) \):
- Neutral Fourier coefficient:
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- Whittaker coefficient:
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Fourier-Jacobi coefficients

- $u := g_1 / (g_1 \cap g^f)$. $\omega_f(X, Y) := \langle f, [X, Y] \rangle$ - symplectic form.
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- $\forall$ isotropic subspace $i \subset u$, let $I := \text{Exp}(i)(A)$

$$\mathcal{F}_{H, f}[\eta](g) := \int_{[I]} \mathcal{F}_{H, f}[\eta](ug) \, du$$

Cf. $\theta$, Stone-von-Neumann thm, Poisson summation formula.
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\[
\mathcal{F}_{H, f}^l [\eta] (g) := \int_{[l]} \mathcal{F}_{H, f} [\eta] (ug) \, du
\]

Lemma (Root exchange, cf. Ginzburg-Rallis-Soudry)

- \( \mathcal{F}_{H, f} [\eta] (g) = \sum_{\gamma \in (U/I^\perp)(\mathbb{K})} \mathcal{F}_{H, f}^l [\eta] (\gamma g) \)

- For any isotropic subspace \( j \subset u \) with \( \dim j = \dim i \) and \( j \cap i^\perp = \{0\} \),

\[
\mathcal{F}_{H, f}^j [\eta] (g) = \int_{J(A)} \mathcal{F}_{H, f}^l [\eta] (ug) \, du
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For \( H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \), \( f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \) : \( \begin{pmatrix} 0 & i & \eta \\ \text{0} & \text{0} & j \\ 0 & 0 & 0 \end{pmatrix} \)

Cf. \( \theta \), Stone-von-Neumann thm, Poisson summation formula.
Relating different coefficients

- \( \text{WO}(\eta) := \{ O \in \mathcal{N}(g) \mid \forall \text{ neutral } (h, f) \text{ with } f \in O, \mathcal{F}_{h,f}(\eta) \neq 0 \} \).
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- Say $(H, f) \succ (S, f)$ if $[H, S] = 0$ and $g^f \cap g^H_{\geq 1} \subseteq g^{S-H}_{\geq 0}$. 

Integrals: 

$$
\int_{V(A)} F_{S,f}(\eta)(v) \, dv
$$
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- $f$ is $\mathbb{K}$-distinguished if $\forall$ Levi $l \ni f$ defined over $\mathbb{K}$, $l = g$.

  Equivalently: the semi-simple part of the centralizer $G_f$ is anisotropic
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- Say $(H, f) \succ (S, f)$ if $[H, S] = 0$ and $g^f \cap g^H \subseteq g^S_{-H}$.
- $f$ is $\mathbb{IK}$-distinguished if $\forall$ Levi $l \ni f$ defined over $\mathbb{IK}$, $l = g$.
  Equivalently: the semi-simple part of the centralizer $G_f$ is anisotropic.
- $(S, f)$ is called Levi-distinguished if $\exists$ parabolic $p = lu$ s.t. $f$ is $\mathbb{IK}$-distinguished in $l$, and $n_{S,f} = l_{S,f} \oplus u$. 

Whittaker coefficients are Levi-distinguished. For Whittaker pairs with the same $f$ and commuting $H$-s, neutral $\succ$ any $\succ$ Levi-distinguished.

Theorem

Let $(H, f) \succ (S, f)$. Then

(i) $F_{H,f}[\eta]$ linearly determines $F_{S,f}[\eta]$.

(ii) If $\Gamma f \in \text{WO}_{\text{max}}(\eta)$ and $g^H_1 = g^S_1 = 0$, let $v^1 : g^H > 1 \cap g^S < 1$. Then $F_{H,f}[\eta](g) = \int V(A) F_{S,f}[\eta](vg) dv$. 


Relating different coefficients

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- Say \((H, f) \succ (S, f)\) if \([H, S] = 0\) and \(g^f \cap g^H \subseteq g^{S-H} \).
- \(f\) is \(\mathbb{I}K\)-distinguished if \(\forall \text{ Levi } l \ni f \text{ defined over } \mathbb{I}K, l = g\).
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- \((S, f)\) is called Levi-distinguished if \(\exists \text{ parabolic } \mathfrak{p} = \mathfrak{l}u\)
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- Say \((H, f) \succ (S, f)\) if \([H, S] = 0\) and \(\mathfrak{g}^f \cap \mathfrak{g}^H_{\geq 1} \subseteq \mathfrak{g}^S_{\geq 0-H}\).
- \(f\) is \(\mathbb{K}\)-distinguished if \(\forall \text{ Levi } l \ni f\) defined over \(\mathbb{K}\), \(l = \mathfrak{g}\).
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**Theorem**

**Let** \((H, f) \succ (S, f)\). Then

1. \(\mathcal{F}_{H,f}[\eta]\) linearly determines \(\mathcal{F}_{S,f}[\eta]\).
2. If \(\Gamma f \in \text{WO}^{\max}(\eta)\) and \(\mathfrak{g}^H_1 = \mathfrak{g}^S_1 = 0\) let \(v := \mathfrak{g}^H_{>1} \cap \mathfrak{g}^S_{<1}\). Then

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\mathcal{F}_{H,f}[\eta](g) = \int_{V(A)} \mathcal{F}_{S,f}[\eta](vg) \, dv
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$$F_{H, f}[\eta](g) = \int_{V(A)} F_{S, f}[\eta](vg) \, dv$$
Theorem

Let \((H, f) \succ (S, f)\). Then

(i) \(F_{H,f}[\eta]\) linearly determines \(F_{S,f}[\eta]\).

(ii) If \(\Gamma f \in WO^{\text{max}}(\eta)\) and \(g_1^H = g_1^S = 0\) let \(v := g_{>1}^H \cap g_{<1}^S\). Then

\[
F_{H,f}[\eta](g) = \int_{V(A)} F_{S,f}[\eta](vg) \, dv
\]

Corollary

If \(\eta\) is cuspidal then any \(O \in WO^{\text{max}}(\eta)\) is \(\mathbb{K}\)-distinguished. In particular, \(O\) is totally even for \(G = \text{Sp}_{2n}\), totally odd for \(G = \text{SO}(V)\), not minimal for \(\text{rk} G > 1\), and not next-to-minimal for \(\text{rk} G > 2\), \(G \neq F_4\).

Lower bounds for partitions of \(O \in WO^{\text{max}}(\eta)\) with cuspidal \(\eta\): 2\(^n\) for \(\text{Sp}_{2n}\), 3\(^n1^n\) for \(\text{SO}(2n, 2n)\), 53\(^{n-1}1^n\) for \(\text{SO}(2n + 1, 2n + 1)\), 3\(^n1^{n+1}\) for \(\text{SO}(2n + 1, 2n)\), and \((3^{n+1}, 1^n)\) for \(\text{SO}(2n + 2, 2n + 1)\).

If \(f \notin WO(\eta)\) then \(F_{H,f}(\eta) = 0\) for any \(H\).
Corollary

(i) If \( \eta \) is cuspidal then any \( O \in WO^{\text{max}}(\eta) \) is \( \mathbb{K} \)-distinguished. In particular, \( O \) is totally even for \( G = \text{Sp}_{2n} \), totally odd for \( G = \text{SO}(V) \), not minimal for \( \text{rk} G > 1 \), and not next-to-minimal for \( \text{rk} G > 2 \), \( G \neq F_4 \).

(ii) Lower bounds for partitions of \( O \in WO^{\text{max}}(\eta) \) with cuspidal \( \eta \):
- \( 2^n \) for \( \text{Sp}_{2n} \),
- \( 3^n1^n \) for \( \text{SO}(2n, 2n) \),
- \( 53^{n-1}1^n \) for \( \text{SO}(2n + 1, 2n + 1) \),
- \( 3^n1^{n+1} \) for \( \text{SO}(2n + 1, 2n) \), and
- \( (3^{n+1}, 1^n) \) for \( \text{SO}(2n + 2, 2n + 1) \).

(iii) If \( f \notin WO(\eta) \) then \( \mathcal{F}_{H,f}(\eta) = 0 \) for any \( H \).

Proof of (i).

Let \( \mathfrak{l} \subset \mathfrak{g} \) be Levi subalgebra intersecting \( O \). Let \((e, h, f) \in \mathfrak{l}\) be an \( \mathfrak{sl}_2 \)-triple with \( f \in \mathcal{O} \). Let \( Z \in \mathfrak{g} \) be a (rational) semi-simple element s.t. \( \mathfrak{l} = \mathfrak{g}^Z \). Let \( T >> 0 \in \mathbb{Z} \) and let \( H := h + TZ \).

Then \( \mathcal{F}_{H,f}(\eta) = \mathcal{F}_{H,f}(c_L(\eta)) \), where \( c_L(\eta) \) denotes the constant term. Since \( \mathcal{F}_{H,f}(\eta) \neq 0 \) by the theorem and \( \eta \) is cuspidal, \( L = G \).
Example for the proof of the Theorem

\[ G := GL(4, \mathbb{A}), \ f := E_{21} + E_{43}, \ H := \text{diag}(3, 1, -1, -3), \\
\ h = \text{diag}(1, -1, 1, -1), \ Z = H - h = \text{diag}(2, 2, -2, -2), \ H_t := h + tZ. \]
Example for the proof of the Theorem

$G := \text{GL}(4, \mathbb{A})$, $f := E_{21} + E_{43}$, $H := \text{diag}(3, 1, -1, -3)$, $h = \text{diag}(1, -1, 1, -1)$, $Z = H - h = \text{diag}(2, 2, -2, -2)$, $H_t := h + tZ$.

Then $n_0 \subset n_{1/4} \oplus i \sim n_{1/4} \oplus j \subset n_{3/4} = n_1$:

\[
\begin{pmatrix}
  0 & * & 0 & * \\
  0 & 0 & 0 & 0 \\
  0 & * & 0 & * \\
  0 & 0 & 0 & 0
\end{pmatrix}
\subset
\begin{pmatrix}
  0 & * & a & * \\
  0 & 0 & 0 & a \\
  0 & - & 0 & * \\
  0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
  0 & * & - & * \\
  0 & 0 & 0 & - \\
  0 & 0 & 0 & * \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
  0 & * & * & * \\
  0 & 0 & - & * \\
  0 & 0 & 0 & * \\
  0 & 0 & 0 & 0
\end{pmatrix}
\subset
\begin{pmatrix}
  0 & * & * & * \\
  0 & 0 & * & * \\
  0 & 0 & 0 & * \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

Both $*$ and $-$ denote arbitrary elements. $*$ denotes the entries in $g^{H_t}_{1}$ and $-$ those in $g^{H_t}_{1}$. $a$ denotes equal elements in $g^{H_t}_{1} \cap g^{f}$. 
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in WO^{\text{max}}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $F_{H, f}[\eta]$ using the left regular action.
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in \text{WO}^{\text{max}}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $\mathcal{F}_{H, f}[\eta]$ using the left regular action.

Proof.

Want to show that $\mathcal{F}_{H, f}[\eta - \eta^u] = 0$. By the theorem, enough to show $\mathcal{F}_{H+Z, f}[\eta - \eta^u] = 0$ for some $Z$. Find $Z$ such that $u \in N_{H+Z, f}$. \qed
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in \text{WO}_{\max}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $F_{H, f}[\eta]$ using the left regular action.

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Example: $G = GL_4(\mathbb{A}), f = E_{31} + E_{42}, H = \text{diag}(1, 1, -1, -1), u = \text{Id} + E_{12} + E_{34}, Z = \text{diag}(1, -1, 1, -1)$.

\[
\begin{pmatrix}
0 & b & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & b \\
0 & 0 & 0 & 0
\end{pmatrix} , \begin{pmatrix}
0 & * & * & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \leftrightarrow \begin{pmatrix}
0 & 0 & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

*: non-zero pairing with $f$. $b$: entries of $u$. 

**Corollary (Hidden symmetry)**

Let \( \eta \) be an automorphic form on \( G \), and let \((H, f)\) be a Whittaker pair with \( \Gamma f \in \text{WO}^\text{max}(\eta) \). Then any unipotent element \( u \) of the centralizer of the pair \((H, f)\) in \( G \) acts trivially on the Fourier coefficient \( \mathcal{F}_{H, f}[\eta] \) using the left regular action.

**Proof.**

Want to show that \( \mathcal{F}_{H, f}[\eta - \eta^u] = 0 \). By the theorem, enough to show \( \mathcal{F}_{H+Z, f}[\eta - \eta^u] = 0 \) for some \( Z \). Find \( Z \) such that \( u \in N_{H+Z, f} \).

Example: \( G = GL_4(\mathbb{A}), f = E_{31} + E_{42}, H = \text{diag}(1, 1, -1, -1), u = Id + E_{12} + E_{34}, Z = \text{diag}(1, -1, 1, -1) \).

\[
\begin{pmatrix}
0 & b & \ast & \ast \\
0 & 0 & \ast & \ast \\
0 & 0 & 0 & b \\
0 & 0 & 0 & 0
\end{pmatrix}
,\quad
\begin{pmatrix}
0 & \ast & \ast & \ast \\
0 & 0 & 0 & \ast \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
0 & 0 & \ast & \ast \\
0 & 0 & \ast & \ast \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( \ast \): non-zero pairing with \( f \). \( b \): entries of \( u \).

Corollary: if \( \text{WO}^\text{max}(\eta) = \{2^n\} \) then \( \mathcal{F}_{H, f}[\eta] \) is Eulerian (Shalika model).
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in \text{WO}^\text{max}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $\mathcal{F}_{H,f}[\eta]$ using the left regular action.

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Want to show that $\mathcal{F}_{H,f}[\eta - \eta^u] = 0$. By the theorem, enough to show $\mathcal{F}_{H+Z,f}[\eta - \eta^u] = 0$ for some $Z$. Find $Z$ such that $u \in N_{H+Z,f}$.

Corollary

If $G = \text{GL}_n(\mathbb{A})$ and $\text{WO}^\text{max}(\eta) = \{2^n\}$ or $G \in \{\text{SO}(n, n), \text{SO}(n + 1, n)\}$ and $\text{WO}^\text{max}(\eta) = \{31 \ldots 1\}$ then $\mathcal{F}_{H,f}[\eta]$ is Eulerian.

Follows from uniqueness of Shalika and Bessel models.
Fourier-Jacobi periods and the Weil representation

For a symplectic space $V$ over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\tilde{J}(V) := \text{Sp}(V(\mathbb{A})) \rtimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation $\varpi_V$ with central character $\chi$. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(\mathbb{K})} \omega_{\chi}(g)f(a), \text{ where } g \in \tilde{J}(V), f \in S(\mathcal{E}(\mathbb{A})), \mathcal{E} \subset V \text{ Lagrangian.}$$
Fourier-Jacobi periods and the Weil representation

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$$\theta_f(g) = \sum_{a \in E(K)} \omega_\chi(g) f(a),$$

where $g \in \tilde{J}(V)$, $f \in S(\mathcal{E}(A))$, $\mathcal{E} \subset V$ Lagrangian.

For a Whittaker pair $(H, f)$ let $u := g^H_{>1}$ and $V := u/n_{H,f}$, with symplectic form $\omega_f(A, B) := \langle f, [A, B] \rangle$. Then we have a natural map $\ell : U \times G_{H,f} \to \tilde{J}(V)$. Define $FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash G_{H,f})$ by

$$f \otimes \eta \mapsto \int_{U(K) \backslash U(A)} f(u\tilde{g}) \theta_\eta(\ell(u, \tilde{g})) \, du$$

$M:=\text{split semi-simple part of the centralizer } G_{H,f}$. 
Fourier-Jacobi periods and the Weil representation

For a symplectic space $V$ over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\tilde{J}(V) := \text{Sp}(V(\mathbb{A})) \ltimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation $\varpi_V$ with central character $\chi$. It has automorphic realization given by theta functions:

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For a Whittaker pair $(H, f)$ let $u := g_{\geq 1}^H$ and $V := u/n_{H, f}$, with symplectic form $\omega_f(A, B) := \langle f, [A, B] \rangle$. Then we have a natural map $\ell : U \times \tilde{G}_{H, f} \to \tilde{J}(V)$. Define $FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash \tilde{G}_{H, f})$ by

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$M :=$ split semi-simple part of the centralizer $G_{H, f}$.

**Theorem**

If $\Gamma \cdot f \in WO^{\text{max}}(\pi)$ then $\tilde{M}$ acts on the image of $FJ$ by $\pm 1$. 
Fourier-Jacobi periods and the Weil representation

For a Whittaker pair \((H, f)\) let \(u := g_{\geq 1}^H\) and \(V := u/n_{H,f}\), with symplectic form \(\omega_\varphi(A, B) := \langle f, [A, B] \rangle\). Then we have a natural map \(\ell : U \rtimes \tilde{G}_\gamma \to J(V)\). Define \(FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \setminus \tilde{G}_{H,f})\) by

\[
f \otimes \eta \mapsto \int_{U(K) \setminus U(A)} f(u\tilde{g})\theta_\eta(\ell(u, \tilde{g})) du
\]

\(M :=\) split semi-simple part of the centralizer \(G_{H,f}\).

**Theorem**

*If \(\Gamma \cdot f \in WO^{\max}(\pi)\) then \(\tilde{M}\) acts on the image of \(FJ\) by \(\pm 1\).*

Since the Weil representation \(\varpi_V\) is genuine, obtain:

**Corollary**

*If \(\Gamma \cdot f \in WO^{\max}(\pi)\) then the cover \(\tilde{M}\) splits.*
Fourier-Jacobi periods and the Weil representation

For a Whittaker pair \((H, f)\) let \(u := g^H_{\geq 1}\) and \(V := u/n_{H,f}\), with symplectic form \(\omega_{\varphi}(A, B) := \langle f, [A, B] \rangle\). Then we have a natural map \(\ell : U \rtimes \tilde{G}_\gamma \to \tilde{J}(V)\). Define \(FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash \tilde{G}_{H,f})\) by

\[
    f \otimes \eta \mapsto \int_{U(K) \backslash U(A)} f(u\tilde{g})\theta_\eta(\ell(u, \tilde{g})) \, du
\]

\(M := \text{split semi-simple part of the centralizer } G_{H,f}\).

**Theorem**

If \(\Gamma \cdot f \in WO^{\max}(\pi)\) then \(\tilde{M}\) acts on the image of \(FJ\) by \(\pm 1\).

Since the Weil representation \(\varpi_V\) is genuine, obtain:

**Corollary**

If \(\Gamma \cdot f \in WO^{\max}(\pi)\) then the cover \(\tilde{M}\) splits.

**Corollary**

If \(\Gamma \cdot f \in WO^{\max}(\pi)\) and \(G\) is classical then the orbit of \(f\) is special.
Eulerianity

Lemma

Let \((S, f)\) and \((H, f')\) be two Whittaker pairs such that 
\(\Gamma f = \Gamma f' \in WO^{\text{max}}(\eta)\). Suppose that a Fourier–Jacobi coefficient \(F_{S,f}^I[\eta]\) is Eulerian. Then any Fourier–Jacobi coefficient \(F_{H,\psi}^{I'}[\eta]\) is also Eulerian.
Lemma

Let \((S, f)\) and \((H, f')\) be two Whittaker pairs such that \(\Gamma f = \Gamma f' \in \text{WO}^{\text{max}}(\eta)\). Suppose that a Fourier–Jacobi coefficient \(\mathcal{F}_{S,f}^I[\eta]\) is Eulerian. Then any Fourier–Jacobi coefficient \(\mathcal{F}_{H,\psi}^{I'}[\eta]\) is also Eulerian.

Question

Is any Fourier–Jacobi coefficient \(\mathcal{F}_{S,f}^I[\eta]\) with \(\Gamma f \in \text{WO}^{\text{max}}(\eta)\) Eulerian for any spherical \(\eta\) that generates an irreducible representation?
Lemma

Let \((S, f)\) and \((H, f')\) be two Whittaker pairs such that \(\Gamma f = \Gamma f' \in \text{WO}^{\text{max}}(\eta)\). Suppose that a Fourier–Jacobi coefficient \(F^l_{S, f}[\eta]\) is Eulerian. Then any Fourier–Jacobi coefficient \(F^l_{H, \psi}[\eta]\) is also Eulerian.

Question

Is any Fourier–Jacobi coefficient \(F^l_{S, f}[\eta]\) with \(\Gamma f \in \text{WO}^{\text{max}}(\eta)\) Eulerian for any spherical \(\eta\) that generates an irreducible representation?

Verified for:

1. Discrete spectrum of \(\text{GL}_n(\mathbb{A})\).
2. Minimal representations of most split simply-laced groups
3. Next-to-minimal Eisenstein series of most split simply-laced groups
Expressing forms through their Whittaker coefficients

Theorem

Any $\mathcal{F}_{H,f}$ is linearly determined by all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

Corollary

(i) Any $\eta$ is linearly determined by all its Levi-distinguished Fourier coefficients.

(ii) If all $O \in \mathcal{W}O(\eta)$ admit Whittaker coefficients then $\eta$ is linearly determined by its Whittaker coefficients.

(iii) If $G$ is split and simply-laced, and $\eta$ is minimal or next-to-minimal then all Fourier coefficients of $\eta$ are linearly determined by Whittaker coefficients.
Theorem

Any $F_{H,f}$ is linearly determined by all Levi-distinguished Fourier coefficients $F_{S,F}$ with $\Gamma F \geq \Gamma f$.

Corollary

(i) Any $\eta$ is linearly determined by all its Levi-distinguished Fourier coefficients.

(ii) If all $O \in \mathcal{W}(\eta)$ admit Whittaker coefficients then $\eta$ is linearly determined by its Whittaker coefficients.

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Expressing forms through their Whittaker coefficients

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- Any $\eta$ is linearly determined by all its Levi-distinguished Fourier coefficients.
- If all $\mathcal{O} \in \text{WO}(\eta)$ admit Whittaker coefficients then $\eta$ is linearly determined by its Whittaker coefficients.
Theorem

Any \( \mathcal{F}_{H,f} \) is linearly determined by all Levi-distinguished Fourier coefficients \( \mathcal{F}_{S,F} \) with \( \Gamma F \geq \Gamma f \).

Corollary

1. Any \( \eta \) is linearly determined by all its Levi-distinguished Fourier coefficients.
2. If all \( O \in \text{WO}(\eta) \) admit Whittaker coefficients then \( \eta \) is linearly determined by its Whittaker coefficients.
3. If \( G \) is split and simply-laced, and \( \eta \) is minimal or next-to-minimal then all Fourier coefficients of \( \eta \) are linearly determined by Whittaker coefficients.
Let $\eta \in C^\infty(\Gamma \backslash \text{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\text{GL}_{n-1}(\mathbb{K})$.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
Explanation for $\text{GL}_n$

Let $\eta \in C^\infty(\Gamma \backslash \text{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\text{GL}_{n-1}(\mathbb{K})$.

Conjugate, restrict to the next column and continue

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
= \cdots
\]
Explanation for $\text{GL}_n$

Let $\eta \in C^\infty(\Gamma \backslash \text{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\text{GL}_{n-1}(\mathbb{K})$. Conjugate, restrict to the next column and continue

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \ldots
$$

$$
\begin{bmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
+ 
\begin{bmatrix}
0 & * & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \ldots
$$
Example: $\text{Sp}(4)$

\[
\text{sp}_4 = \left\{ \left( \begin{array}{cc} A & B = B^t \\ C = C^t & -A^t \end{array} \right) \right\}.
\]

Let $\mathfrak{n}$ be the Borel nilradical, and $u \subset \mathfrak{n}$ be the Siegel nilradical, spanned by $B$. Characters given by $\bar{u} \cong \text{Sym}^2(\mathbb{K}^2)$. Restricting $\eta$ to $B$ and decomposing into Fourier series we obtain $\eta = \sum_{f \in \bar{u}} F_{u,f}[\eta]$.

1. **Constant term $F_{u,0}[\eta]$:** Restrict to the Siegel Levi $L \cong \text{GL}_2(\mathbb{A})$, and decompose to Fourier series on the abelian group $N \cap L$:

\[
F_{u,0}[\eta] = \sum_{a \in \mathbb{K}} W_{a,0}[\eta].
\]
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\[
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F_{u,0}[\eta] = \sum_{a \in \mathbb{K}} W_{a,0}[\eta].
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2. **Any $f$ of rank one is conjugate under $L$ to $f_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.**

Decomposing $F_{u,f_1}[\eta]$ on $N \cap L$:

\[
F_{u,f_1}[\eta] = \sum_{a \in \mathbb{K}} W_{a,1}[\eta].
\]
\[ \mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\} ; \quad \eta = \sum_{f \in \mathfrak{u}} \mathcal{F}_{u,f} [\eta]. \]

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   \[
   \mathcal{F}_{u,f_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,1}[\eta].
   \]

3. **Split non-degenerate forms** are conjugate to \( f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
   Using Weyl group conjugation (24) and root exchange, we express \( \mathcal{F}_{u,f_2} \) through \( \mathcal{F}_{u',e_{21}} \), where \( u' = \text{Span}(e_{12} - e_{43}, e_{13}, e_{24}) \subset \mathfrak{n} \).
   Fourier expansion by the remaining coordinate of \( e_{14} + e_{23} \in \mathfrak{n} \):
   \[
   \mathcal{F}_{u,f}[\eta](g) = \int_{\chi \in \mathcal{A}} \mathcal{W}_{1,a}[\eta](\chi) g \chi d\chi.
   \]
$X$ is the set of anisotropic $2 \times 2$ forms. For $f \in X$, we cannot simplify $\mathcal{F}_{u,f}[\eta]$. Summarizing, for any $\eta$ on $G = \text{Sp}_4(\mathbb{A})$ we have

$$
\eta(g) = \sum_{f \in X} \mathcal{F}_{u,f}[\eta](g) + \sum_{a \in K} \left( \sum_{\gamma \in L/O(1,1)} \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta](v_x w \gamma g) + \sum_{\gamma \in L/(N \cap L)} \mathcal{W}_{a,1}[\eta](\gamma g) + \mathcal{W}_{a,0}[\eta](g) \right)
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3. If \(\eta\) is cuspidal and non-generic then \(\eta = \sum_{f \in X} F_{u,f}[\eta]\).
$F := \text{a } p\text{-adic local field, } G := \mathbf{G}(F), \ g := \text{Lie}(G), \ \forall$ smooth representation $\pi$,
- $F := \text{a } p\text{-adic local field, } G := \mathbf{G}(F), \ g := \text{Lie}(G),$
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- $u/\mathfrak{n}_{H, f}$ is a symplectic space, and its Heisenberg group $\mathcal{H}$ is a quotient of $U$.
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- $\omega_{H,f} :=$ oscillator representation of $\mathcal{H}$ lifted to $u$. $\mathcal{W}_{H,f} := \text{ind}_U^G \omega_{H,f}$
- For all smooth representation $\pi$, define its $(H, f)$-Whittaker quotient by
  \[
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- All the theorems above have local analogues with similar proofs.
Wave front set and wave-front cycle

Let $\pi$ be smooth, admissible and finitely generated.

**Theorem (Howe, Harish-Chandra 70s)**

Near $e \in G$, the character distribution equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

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**Theorem (Moeglin-Waldspurger, 87')**

- If $\pi_{H,f} \neq 0$ then $f \in WF(\pi)$. 
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- If $\pi_{H,f} \neq 0$ then $f \in WF(\pi)$.
- If $f \in WF^{\text{max}}(\pi)$ then $\dim \pi_{H,f} = c_f$. 