Relations between Fourier coefficients of automorphic forms with applications to vanishing and Eulerianity

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Definitions

- $K$: number field, $A := A_K$, $G$: reductive group over $K$, $\Gamma := G(K)$, $G := G(A)$, $g := \text{Lie}(\Gamma)$.

Fix a semisimple $H \in g$, and let $g_i := g H_i$ denote the eigenspaces of $\text{ad}(H)$. Assume that all the eigenvalues $i$ lie in $Q$.

Let $f \in g - 2$. Call $(H, f) \in g \times g$ a Whittaker pair.

Define $n := n H, f := (g_1 \cap g_f) \oplus \bigoplus_{i > 1} g_i$, $N := \text{Exp}(n)(A)$.

Fix a non-trivial unitary additive character $\psi: K \setminus A \to \mathbb{C}$ and define $\chi_f: N \to \mathbb{C}$ by $\chi_f(\text{Exp}(X)) := \psi(\langle f, X \rangle)$.

Let $[N] := (\Gamma \cap N) \setminus N$. For automorphic form $\eta$ on $G$, define Fourier coefficient $F_{H, f}[\eta](g) := \int_{[N]} \eta(ng) \chi_f(n)^{-1} dn$. 

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Fourier coefficients

February 2021 2 / 24
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- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i := \mathfrak{g}_i^H$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues $i$ lie in $\mathbb{Q}$.

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- Let $f \in \mathfrak{g}_{-2}$. Call $(H, f) \in \mathfrak{g} \times \mathfrak{g}$ a Whittaker pair.
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- Define $n := n_{H,f} := (g_1 \cap g^f) \oplus \bigoplus_{i > 1} g_i$, $N := \text{Exp}(n)(A)$. 

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- Fix a semisimple $H \in \mathfrak{g}$, and let $\mathfrak{g}_i := \mathfrak{g}_i^H$ denote the eigenspaces of $ad(H)$. Assume that all the eigenvalues $i$ lie in $\mathbb{Q}$.
- Let $f \in \mathfrak{g}_{-2}$. Call $(H, f) \in \mathfrak{g} \times \mathfrak{g}$ a Whittaker pair.
- Define $\mathfrak{n} := \mathfrak{n}_{H,f} := (\mathfrak{g}_1 \cap \mathfrak{g}_f^F) \oplus \bigoplus_{i>1} \mathfrak{g}_i$, $N := \text{Exp}(\mathfrak{n})(\mathbb{A})$.
- Fix a non-trivial unitary additive character $\psi : \mathbb{K} \backslash \mathbb{A} \to \mathbb{C}$ and define $\chi_f : N \to \mathbb{C}$ by $\chi_f(\text{Exp} X) := \psi(\langle f, X \rangle)$.
Definitions

- \( \mathbb{K} \): number field, \( \mathbb{A} := \mathbb{A}_\mathbb{K} \), \( \mathbf{G} \): reductive group over \( \mathbb{K} \), \( \Gamma := \mathbf{G}(\mathbb{K}) \), \( G := \mathbf{G}(\mathbb{A}) \), \( \mathfrak{g} := \text{Lie}(\Gamma) \).

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- Define \( n := n_{H,f} := (\mathfrak{g}_1 \cap \mathfrak{g}^f) \oplus \bigoplus_{i > 1} \mathfrak{g}_i \), \( N := \text{Exp}(n)(\mathbb{A}) \).

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- Let \([N] := (\Gamma \cap N) \backslash N\). For automorphic form \( \eta \) on \( G \), define Fourier coefficient

\[
\mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1} \, dn.
\]
Two central cases of Fourier coefficients

\[[H, f] = -2f, \ n = (g_1 \cap g^f) \oplus \bigoplus_{i > 1} g_i, \ N = \text{Exp}(n)(A),\]

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- Neutral Fourier coefficient, coming from \( sl_2 \)-triple (e,H,f), e.g.:

\[
H = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}, \quad 
f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad 
n = \begin{pmatrix}
0 & * & 0 & * \\
0 & 0 & 0 & 0 \\
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\end{pmatrix}
\]

\[
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1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
n = \begin{pmatrix}
0 & \ast & 0 & \ast & \ast \\
0 & 0 & 0 & 0 & 0 \\
0 & \ast & 0 & \ast & \ast \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

- Whittaker coefficient \( \mathcal{W}_f \), with \( N \) maximal unipotent, e.g.:

\[
H = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & -3
\end{pmatrix}
\]

\[
f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
n = \begin{pmatrix}
0 & \ast & \ast & \ast & \ast \\
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0 & 0 & 0 & \ast & \ast \\
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\[ [H, f] = -2f, \quad n = (g_1 \cap g^f) \oplus \bigoplus_{i > 1} g_i, \quad N = \text{Exp}(n)(A), \]

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\end{pmatrix}
\quad f = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad n = \begin{pmatrix}
0 & * & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}
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0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad n = \begin{pmatrix}
0 & * & * & * & * \\
0 & 0 & * & * & 0 \\
0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.
Two central cases of Fourier coefficients

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  \end{pmatrix}
  \quad f = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
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  0 & * & * & * \\
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Coefficients that are both neutral and Whittaker are Eulerian by local uniqueness of Whittaker models.
Examples of Fourier coefficients

\[ [H, f] = -2f, \quad n = (g_1 \cap g^f) \oplus \bigoplus_{i > 1} g_i, \quad N = \text{Exp}(n)(\mathbb{A}) \]

\[ \mathcal{F}_{H,f}[\eta](g) := \int_{[N]} \eta(ng)\chi_f(n)^{-1} dn. \]

Comparison for \( G = \text{GL}_3(\mathbb{A}) \):

- Neutral Fourier coefficient:

\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Examples of Fourier coefficients

\[ [H, f] = -2f, \; n = (g_1 \cap g^f) \oplus \bigoplus_{i > 1} g_i, \; N = \text{Exp}(n)(A) \]

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Comparison for \( G = \text{GL}_3(A) \):

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  0 & 0 & 0 \\
  0 & 0 & 0 \\
  1 & 0 & 0 
  \end{pmatrix},
  n = \begin{pmatrix}
  0 & 0 & * \\
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  \end{pmatrix}
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Fourier-Jacobi coefficients

- $u = g_1 / (g_1 \cap g^f)$. $\omega_f(X, Y) := \langle f, [X, Y] \rangle$ - symplectic form.
Fourier-Jacobi coefficients

- $u := \frac{g_1}{g_1 \cap g^f}$. $\omega_f(X, Y) := \langle f, [X, Y] \rangle$- symplectic form.
- $\forall$ isotropic subspace $i \subset u$, let $I := \text{Exp}(i)(A)$

$$F_{H,f}^I[\eta](g) := \int_{[I]} F_{H,f}[\eta](ug) \, du$$

Cf. $\theta$, Stone-von-Neumann thm, Poisson summation formula.
Fourier-Jacobi coefficients

- $u := g_1/(g_1 \cap g^f)$. $\omega_f(X, Y) := \langle f, [X, Y] \rangle$- symplectic form.
- For any isotropic subspace $i \subset u$, let $l := \text{Exp}(i)(A)$

$$\mathcal{F}_{H,f}^l[\eta](g) := \int_{[l]} \mathcal{F}_{H,f}[\eta](ug) \, du$$
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$$F^l_{H,f} [\eta] (g) := \int I F_{H,f} [\eta] (ug) \, du$$

Lemma (Root exchange, cf. Ginzburg-Rallis-Soudry)

- $F_{H,f} [\eta] (g) = \sum_{\gamma \in (u/I^\perp)(K)} F^l_{H,f} [\eta] (\gamma g)$
- For any isotropic subspace $j \subset u$ with $\dim j = \dim i$ and $j \cap i^\perp = \{0\}$, $j(A)

$$F^J_{H,f} [\eta] (g) = \int J(A) F^l_{H,f} [\eta] (ug) \, du$$
Fourier-Jacobi coefficients

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\mathcal{F}^I_{H,f}[\eta](g) := \int_{[I]} \mathcal{F}_{H,f}[\eta](ug) \, du
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Lemma (Root exchange, cf. Ginzburg-Rallis-Soudry)

- \( \mathcal{F}_{H,f}[\eta](g) = \sum_{\gamma \in (U/I^\perp)(\mathbb{K})} \mathcal{F}^I_{H,f}[\eta](\gamma g) \)
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\[
\mathcal{F}^J_{H,f}[\eta](g) = \int_{J(\mathbb{A})} \mathcal{F}^I_{H,f}[\eta](ug) \, du
\]

For \( H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \), \( f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \): \( \begin{pmatrix} 0 & i & n \\ 0 & 0 & j \\ 0 & 0 & 0 \end{pmatrix} \)

Cf. \( \theta \), Stone-von-Neumann thm, Poisson summation formula.
Relating different coefficients

- $\text{WO}(\eta) := \{ \mathcal{O} \in \mathcal{N}(g) \mid \forall \text{ neutral } (h, f) \text{ with } f \in \mathcal{O}, \mathcal{F}_{h,f}(\eta) \neq 0 \}$. 

Say $(H, f) \succ (S, f)$ if $[H, S] = 0$ and $g(f) \cap g(H) \geq 1 \subseteq g(S) - g(H) \geq 0$.

$f$ is $K$-distinguished if $\forall$ Levi $l \ni f$ defined over $K$, $l = g$.

Equivalently: the semi-simple part of the centralizer $G_f$ is anisotropic $(S, f)$ is called Levi-distinguished if $\exists$ parabolic $p = lu$ s.t. $f$ is $K$-distinguished in $l$, and $n_S, f = l_S, f \oplus u$.

Whittaker coefficients are Levi-distinguished.

For Whittaker pairs with the same $f$ and commuting $H$-s, neutral $\succ$ any $\succ$ Levi-distinguished.

Theorem

Let $(H, f) \succ (S, f)$. Then

(i) $\mathcal{F}_S, f[\eta]$ can be expressed through $\mathcal{F}_H, f[\eta]$.

(ii) If $\Gamma_f \in \text{WO}_\text{max}(\eta)$ and $g(H) = g(S) = 0$ let $v: = g(H) > 1 \cap g(S) < 1$. Then $\mathcal{F}_H, f[\eta](g) = \int_V(A) \mathcal{F}_S, f[\eta](vg) \, dv$. 

Relating different coefficients

- $\text{WO}(\eta) := \{ \mathcal{O} \in \mathcal{N}(g) \mid \forall \text{ neutral } (h, f) \text{ with } f \in \mathcal{O}, \mathcal{F}_{h, f}(\eta) \neq 0 \}$. 
- Say $(H, f) \succ (S, f)$ if $[H, S] = 0$ and $g^f \cap g^H \subseteq g^{S-H}$. 

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Relating different coefficients

- $\text{WO}(\eta) := \{ \mathcal{O} \in \mathcal{N}(\mathfrak{g}) \mid \forall \text{ neutral } (h, f) \text{ with } f \in \mathcal{O}, \mathcal{F}_{h, f}(\eta) \neq 0 \}.$
- Say $(H, f) \succ (S, f)$ if $[H, S] = 0$ and $\mathfrak{g}^f \cap \mathfrak{g}^H_{\geq 1} \subseteq \mathfrak{g}^S_{\geq 0}.$
- $f$ is $\mathbb{K}$-distinguished if $\forall$ Levi $\mathfrak{l} \ni f$ defined over $\mathbb{K}, \mathfrak{l} = \mathfrak{g}$.
  Equivalently: the semi-simple part of the centralizer $G_f$ is anisotropic.
Relating different coefficients

- \( \text{WO}(\eta) := \{ \mathcal{O} \in \mathcal{N}(g) \mid \forall \text{ neutral } (h, f) \text{ with } f \in \mathcal{O}, \mathcal{F}_{h,f}(\eta) \neq 0 \} \).
- Say \((H, f) \succ (S, f)\) if \([H, S] = 0 \text{ and } g^f \cap g^H_{\geq 1} \subseteq g_{\geq 0}^{S-H} \).
- \(f\) is \(\mathbb{I}K\)-distinguished if \(\forall \text{ Levi } l \ni f \text{ defined over } \mathbb{I}K, \ l = g\).
  Equivalently: the semi-simple part of the centralizer \(G_f\) is anisotropic.
- \((S, f)\) is called Levi-distinguished if \(\exists \) parabolic \(p = lu\)
  s.t. \(f\) is \(\mathbb{I}K\)-distinguished in \(l\), and \(n_{S,f} = l_{S,f} \oplus u\).
Relating different coefficients

- \( \text{WO}(\eta) : = \{ O \in \mathcal{N}(g) \mid \forall \text{ neutral } (h, f) \text{ with } f \in O, F_{h,f}(\eta) \neq 0 \} \).
- Say \((H, f) \succ (S, f)\) if \([H, S] = 0\) and \(g^f \cap g^H \subseteq g^{S-H} \).
- \(f\) is \(\mathbb{K}\)-distinguished if \(\forall\) Levi \(l \ni f\) defined over \(\mathbb{K}\), \(l = g\).
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- Say \((H, f) \succ (S, f)\) if \([H, S] = 0\) and \(g_f \cap g^H_{\geq 1} \subseteq g^S_{\geq 0} \).
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**Theorem**

Let \((H, f) \succ (S, f)\). Then

1. \(F_{S,f}[\eta]\) can be expressed through \(F_{H,f}[\eta]\).
2. If \(\Gamma f \in \text{WO}^{\text{max}}(\eta)\) and \(g^H_1 = g^S_1 = 0\) let \(v := g^H_{>1} \cap g^S_{<1}\). Then

\[
F_{H,f}[\eta](g) = \int_{V(A)} F_{S,f}[\eta](vg) \, dv
\]
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Let \((H, f) \succ (S, f)\). Then

1. \(\mathcal{F}_{S,f}[\eta]\) can be expressed through \(\mathcal{F}_{H,f}[\eta]\).

2. If \(\Gamma f \in \text{WO}^{\max}(\eta)\) and \(g^H_1 = g^S_1 = 0\) let \(v := g^H_{>1} \cap g^S_{<1}\). Then

\[
\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg) \, dv
\]
Theorem

Let $(H, f) \succ (S, f)$. Then

- $\mathcal{F}_{S,f}[\eta]$ can be expressed through $\mathcal{F}_{H,f}[\eta]$.

- If $\Gamma f \in \text{WO}^{\text{max}}(\eta)$ and $g_1^H = g_1^S = 0$ let $v := g_{>1}^H \cap g_{<1}^S$. Then

$$\mathcal{F}_{H,f}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S,f}[\eta](vg) \, dv$$

Corollary

- If $\eta$ is cuspidal then any $O \in \text{WO}^{\text{max}}(\eta)$ is $\mathbb{K}$-distinguished. In particular, $O$ is totally even for $G = \text{Sp}_{2n}$, totally odd for $G = \text{SO}(V)$, not minimal for $\text{rk} G > 1$, and not next-to-minimal for $\text{rk} G > 2$, $G \neq F_4$.

- Lower bounds for partitions of $O \in \text{WO}^{\text{max}}(\eta)$ with cuspidal $\eta$:
  - $2^n$ for $\text{Sp}_{2n}$,
  - $3^n1^n$ for $\text{SO}(2n, 2n)$,
  - $53^{n-1}1^n$ for $\text{SO}(2n+1, 2n+1)$,
  - $3^n1^{n+1}$ for $\text{SO}(2n+1, 2n)$,
  - $(3^{n+1}, 1^n)$ for $\text{SO}(2n+2, 2n+1)$.

- If $f \not\in \text{WO}(\eta)$ then $\mathcal{F}_{H,f}(\eta) = 0$ for any $H$. 
Corollary

(i) If $\eta$ is cuspidal then any $O \in \text{WO}^{\text{max}}(\eta)$ is $\mathbb{K}$-distinguished. In particular, $O$ is totally even for $G = \text{Sp}_{2n}$, totally odd for $G = \text{SO}(V)$, not minimal for $\text{rk} G > 1$, and not next-to-minimal for $\text{rk} G > 2$, $G \neq F_4$.

(ii) Lower bounds for partitions of $O \in \text{WO}^{\text{max}}(\eta)$ with cuspidal $\eta$:
- $2^n$ for $\text{Sp}_{2n}$,
- $3^n1^n$ for $\text{SO}(2n, 2n)$,
- $53^{n-1}1^n$ for $\text{SO}(2n + 1, 2n + 1)$,
- $3^n1^{n+1}$ for $\text{SO}(2n + 1, 2n)$, and
- $(3^{n+1}, 1^n)$ for $\text{SO}(2n + 2, 2n + 1)$.

(iii) If $f \notin \text{WO}(\eta)$ then $\mathcal{F}_{H,f}(\eta) = 0$ for any $H$.

Proof of (i).

Let $l \subset g$ be Levi subalgebra intersecting $O$. Let $(e, h, f) \in l$ be an $\mathfrak{sl}_2$-triple with $f \in O$. Let $Z \in g$ be a (rational) semi-simple element s.t. $l = g^Z$. Let $T \gg 0 \in \mathbb{Z}$ and let $H := h + TZ$.

Then $\mathcal{F}_{H,f}(\eta) = \mathcal{F}_{H,f}(c_L(\eta))$, where $c_L(\eta)$ denotes the constant term. Since $\mathcal{F}_{H,f}(\eta) \neq 0$ by the theorem and $\eta$ is cuspidal, $L = G$. 

Dmitry Gourevitch

Fourier coefficients

February 2021 8 / 24
Example for the proof of the Theorem

\[ G := \text{GL}(4, \mathbb{A}), \quad f := E_{21} + E_{43}, \quad H := \text{diag}(3, 1, -1, -3), \]
\[ h = \text{diag}(1, -1, 1, -1), \quad Z = H - h = \text{diag}(2, 2, -2, -2), \quad H_t := h + tZ. \]
Example for the proof of the Theorem

\[ G := \text{GL}(4, \mathbb{A}), \ f := E_{21} + E_{43}, \ H := \text{diag}(3, 1, -1, -3), \]
\[ h = \text{diag}(1, -1, 1, -1), \ Z = H - h = \text{diag}(2, 2, -2, -2), \ H_t := h + tZ. \]

Then \( n_0 \subset n_{1/4} \oplus i \sim n_{1/4} \oplus j \subset n_{3/4} = n_1 : \)

\[
\begin{pmatrix}
0 & * & 0 & * \\
0 & 0 & 0 & 0 \\
0 & * & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\subset
\begin{pmatrix}
0 & * & a & * \\
0 & 0 & 0 & a \\
0 & - & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
0 & * & - & * \\
0 & 0 & 0 & - \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & * & * & * \\
0 & 0 & - & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\subset
\begin{pmatrix}
0 & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Both * and – denote arbitrary elements. * denotes the entries in \( g_1^{H_t} \) and – those in \( g_1^{H_t} \cap g^f \). a denotes equal elements in \( g_1^{H_t} \cap g^f \).
Corollary (Hidden symmetry)

Let \( \eta \) be an automorphic form on \( G \), and let \((H, f)\) be a Whittaker pair with \( \Gamma f \in W^{\text{max}}(\eta) \). Then any unipotent element \( u \) of the centralizer of the pair \((H, f)\) in \( G \) acts trivially on the Fourier coefficient \( F_{H,f}[\eta] \) using the left regular action.
Corollary (Hidden symmetry)

Let \( \eta \) be an automorphic form on \( G \), and let \((H, f)\) be a Whittaker pair with \( \Gamma f \in \text{WO}^{\text{max}}(\eta) \). Then any unipotent element \( u \) of the centralizer of the pair \((H, f)\) in \( G \) acts trivially on the Fourier coefficient \( \mathcal{F}_{H,f}[\eta] \) using the left regular action.

Proof.

Want to show that \( \mathcal{F}_{H,f}[\eta - \eta^u] = 0 \). By the theorem, enough to show \( \mathcal{F}_{H+Z,f}[\eta - \eta^u] = 0 \) for some \( Z \). Find \( Z \) such that \( u \in N_{H+Z,f} \). 

\[ \square \]
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in \text{WO}_{\text{max}}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $F_{H,f}[\eta]$ using the left regular action.

Proof.

Want to show that $F_{H,f}[\eta - \eta^u] = 0$. By the theorem, enough to show $F_{H+Z,f}[\eta - \eta^u] = 0$ for some $Z$. Find $Z$ such that $u \in N_{H+Z,f}$.

Example: $G = GL_4(\mathbb{A}), f = E_{31} + E_{42}, H = \text{diag}(1, 1, -1, -1), u = \text{Id} + E_{12} + E_{34}, Z = \text{diag}(1, -1, 1, -1)$.

$$n_{H,f} = \begin{pmatrix} 0 & b & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}; n_{H+Z,f} = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$: non-zero pairing with $f$. $b$: entries of $u$. 

\[ \text{Corollary: if } \text{WO}_{\text{max}}(\eta) = \{2n\} \text{ then } \eta \text{ has almost-Shalika model.} \]
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in \text{WO}^{\text{max}}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $\mathcal{F}_{H,f}[\eta]$ using the left regular action.

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Want to show that $\mathcal{F}_{H,f}[\eta - \eta^u] = 0$. By the theorem, enough to show $\mathcal{F}_{H+Z,f}[\eta - \eta^u] = 0$ for some $Z$. Find $Z$ such that $u \in N_{H+Z,f}$.

Example: $G = GL_4(A), f = E_{31} + E_{42}, H = \text{diag}(1, 1, -1, -1), u = \text{Id} + E_{12} + E_{34}, Z = \text{diag}(1, -1, 1, -1)$.

\[
\begin{pmatrix}
0 & b & \ast & \ast \\
0 & 0 & \ast & \ast \\
0 & 0 & 0 & b \\
0 & 0 & 0 & 0
\end{pmatrix} ; \begin{pmatrix}
0 & * & * & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

$\ast$: non-zero pairing with $f$. $b$: entries of $u$.

Corollary: if $\text{WO}^{\text{max}}(\eta) = \{2^n\}$ then $\eta$ has almost-Shalika model.
Corollary (Hidden symmetry)

Let $\eta$ be an automorphic form on $G$, and let $(H, f)$ be a Whittaker pair with $\Gamma f \in \text{WO}^{\text{max}}(\eta)$. Then any unipotent element $u$ of the centralizer of the pair $(H, f)$ in $G$ acts trivially on the Fourier coefficient $F_{H, f}[\eta]$ using the left regular action.

Proof.

Want to show that $F_{H, f}[\eta - \eta^u] = 0$. By the theorem, enough to show $F_{H+Z, f}[\eta - \eta^u] = 0$ for some $Z$. Find $Z$ such that $u \in N_{H+Z, f}$. 

Corollary

If $G \in \{SO(n, n), SO(n + 1, n)\}$ and $\text{WO}^{\text{max}}(\eta) = \{31 \ldots 1\}$ then $F_{H, f}[\eta]$ is Eulerian.

Follows from uniqueness of Bessel models.
Fourier-Jacobi periods and the Weil representation

For a symplectic space $V$ over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\tilde{J}(V) := \text{Sp}(V(\mathbb{A})) \ltimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation $\varpi_V$ with central character $\chi$. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(K)} \omega(\chi(g))f(a),$$

where $g \in \tilde{J}(V)$, $f \in S(\mathcal{E}(\mathbb{A}))$, $\mathcal{E} \subset V$ Lagrangian.
Fourier-Jacobi periods and the Weil representation

For a symplectic space $V$ over $\mathbb{K}$, let $\mathcal{H}(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\tilde{J}(V) := \text{Sp}(V(\mathbb{A})) \rtimes \mathcal{H}(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation $\varpi_V$ with central character $\chi$. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(K)} \omega_{\chi}(g)f(a), \text{ where } g \in \tilde{J}(V), f \in S(\mathcal{E}(\mathbb{A})), \mathcal{E} \subset V \text{ Lagrangian}$$

For a Whittaker pair $(H, f)$ let $u := g_{\geq 1}^H$ and $V := u/\text{n}_{H,f}$, with symplectic form $\omega_f(A, B) := \langle f, [A, B] \rangle$. Then we have a natural map $\ell: U \times \tilde{G}_{H,f} \to \tilde{J}(V)$. Define $FJ: \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash \tilde{G}_{H,f})$ by

$$f \otimes \eta \mapsto \int_{U(K) \backslash U(\mathbb{A})} f(u\tilde{g})\theta_\eta(\ell(u, \tilde{g})) du$$

$M:=\text{split semi-simple part of the centralizer } G_{H,f}$. 
Fourier-Jacobi periods and the Weil representation

For a symplectic space $V$ over $\mathbb{K}$, let $H(V) := V \oplus \mathbb{K}$ be the Heisenberg group and $\tilde{J}(V) := \text{Sp}(V(\mathbb{A})) \ltimes H(V(\mathbb{A}))$ be the double cover Jacobi group. It has unique irreducible unitarizable representation $\varpi_V$ with central character $\chi$. It has automorphic realization given by theta functions:

$$\theta_f(g) = \sum_{a \in \mathcal{E}(K)} \omega_{\chi}(g)f(a), \text{ where } g \in \tilde{J}(V), f \in S(\mathcal{E}(\mathbb{A})), \mathcal{E} \subset V \text{ Lagrangian.}$$

For a Whittaker pair $(H, f)$ let $u := g^{H}_{\geq 1}$ and $V := u/n_{H,f}$, with symplectic form $\omega_f(A, B) := \langle f, [A, B] \rangle$. Then we have a natural map $\ell : U \ltimes \tilde{G}_{H,f} \to \tilde{J}(V)$. Define $FJ : \pi \otimes \varpi_V \to C^\infty(\Gamma \backslash \tilde{G}_{H,f})$ by

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$M$ := split semi-simple part of the centralizer $G_{H,f}$.

Theorem

If $\Gamma \cdot f \in WO_{\text{max}}(\pi)$ then $\tilde{M}$ acts on the image of $FJ$ by $\pm 1.$
Fourier-Jacobi periods and the Weil representation

For a Whittaker pair \((H, f)\) let \(u := g^H_{\geq 1}\) and \(V := u/n_{H,f}\), with symplectic form \(\varphi(A, B) := \langle f, [A, B] \rangle\). Then we have a natural map \(\ell : U \times \tilde{G}_{\gamma} \to \tilde{J}(V)\). Define \(FJ : \pi \otimes \varpi_{V} \to C^\infty(\Gamma \backslash \tilde{G}_{H,f})\) by

\[
    f \otimes \eta \mapsto \int_{U(K) \backslash U(A)} f(u\tilde{g})\theta_{\eta}(\ell(u, \tilde{g})) du
\]

\(M :=\) split semi-simple part of the centralizer \(G_{H,f}\).

**Theorem**

*If \(\Gamma \cdot f \in \text{WO}^{\max}(\pi)\) then \(\tilde{M}\) acts on the image of \(FJ\) by \(\pm 1\).*

Since the Weil representation \(\varpi_{V}\) is genuine, obtain:

**Corollary**

*If \(\Gamma \cdot f \in \text{WO}^{\max}(\pi)\) then the cover \(\tilde{M}\) splits.*
Fourier-Jacobi periods and the Weil representation

For a Whittaker pair \((H, f)\) let \(u := g_H^H\) and \(V := u/n_{H,f}\), with symplectic form \(\omega_\varphi(A, B) := \langle f, [A, B] \rangle\). Then we have a natural map \(\ell : U \rtimes \tilde{G}_\gamma \rightarrow J(V)\). Define \(FJ : \pi \otimes \varpi_V \rightarrow C^\infty(\Gamma \backslash \tilde{G}_{H,f})\) by

\[
f \otimes \eta \mapsto \int_{U(K) \backslash U(A)} f(u\tilde{g}) \theta_\eta(\ell(u, \tilde{g})) du
\]

\(M := \text{split semi-simple part of the centralizer } G_{H,f}\).

**Theorem**

If \(\Gamma \cdot f \in W_{\text{max}}(\pi)\) then \(\tilde{M}\) acts on the image of \(FJ\) by \(\pm 1\).

Since the Weil representation \(\varpi_V\) is genuine, obtain:

**Corollary**

If \(\Gamma \cdot f \in W_{\text{max}}(\pi)\) then the cover \(\tilde{M}\) splits.

**Corollary**

If \(\Gamma \cdot f \in W_{\text{max}}(\pi)\) and \(G\) is classical then the orbit of \(f\) is special.
Lemma

Let \((S, f)\) and \((H, f')\) be two Whittaker pairs such that 
\[ \Gamma f = \Gamma f' \in WO^{\text{max}}(\eta). \]
Let \(I, I'\) be maximal isotropic. Suppose that 
\[ \mathcal{F}_{S,f}^I[\eta] \] is Eulerian. Then 
\[ \mathcal{F}_{H,\psi}^{I'}[\eta] \] is also Eulerian.
Eulerianity

Lemma

Let \((S, f)\) and \((H, f')\) be two Whittaker pairs such that \(\Gamma f = \Gamma f' \in WO^{\text{max}}(\eta)\). Let \(I, I'\) be maximal isotropic. Suppose that \(\mathcal{F}^I_{S,f}[\eta]\) is Eulerian. Then \(\mathcal{F}^{I'}_{H,\psi}[\eta]\) is also Eulerian.

Question

Is any Fourier-Jacobi coefficient \(\mathcal{F}^I_{S,f}[\eta]\) with \(\Gamma f \in WO^{\text{max}}(\eta)\) and \(I\) maximal isotropic Eulerian for any spherical \(\eta\) that generates an irreducible representation?
**Lemma**

Let \((S, f)\) and \((H, f')\) be two Whittaker pairs such that 
\[\Gamma f = \Gamma f' \in WO_{\text{max}}(\eta).\]
Let \(I, I'\) be maximal isotropic. Suppose that 
\[F_{I, S,f}[\eta]\] is Eulerian. Then 
\[F_{I', H,\psi}[\eta]\] is also Eulerian.

**Question**

Is any Fourier-Jacobi coefficient 
\[F_{I, S,f}[\eta]\] with \(\Gamma f \in WO_{\text{max}}(\eta)\) and \(I\) maximal isotropic Eulerian for any spherical \(\eta\) that generates an irreducible representation?

Verified for:

1. Minimal representations of most split simply-laced groups
2. Next-to-minimal Eisenstein series of most split simply-laced groups
3. Discrete spectrum of \(GL_n(\mathbb{A})\).
Theorem

Any $\mathcal{F}_{H,f}$ can be expressed through all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$. 

Corollary

(i) Any $\eta$ can be expressed through all its Levi-distinguished Fourier coefficients.

(ii) If all $O \in W_0(\eta)$ admit Whittaker coefficients then $\eta$ can be expressed through its Whittaker coefficients.

(iii) For split simply-laced $G$, we obtained expressions for all minimal or next-to-minimal $\eta$, and all their Fourier coefficients in terms of Whittaker coefficients.
Theorem

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Corollary

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**Theorem**

Any $\mathcal{F}_{H,f}$ can be expressed through all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

**Corollary**

(i) Any $\eta$ can be expressed through all its Levi-distinguished Fourier coefficients.

(ii) If all $O \in \text{WO}(\eta)$ admit Whittaker coefficients then $\eta$ can be expressed through its Whittaker coefficients.
Expressing forms through their Whittaker coefficients

Theorem

Any $\mathcal{F}_{H,f}$ can be expressed through all Levi-distinguished Fourier coefficients $\mathcal{F}_{S,F}$ with $\Gamma F \geq \Gamma f$.

Corollary

(i) Any $\eta$ can be expressed through all its Levi-distinguished Fourier coefficients.

(ii) If all $\mathcal{O} \in \text{WO}(\eta)$ admit Whittaker coefficients then $\eta$ can be expressed through its Whittaker coefficients.

(iii) For split simply-laced $G$, we obtained expressions for all minimal or next-to-minimal $\eta$, and all their Fourier coefficients in terms of Whittaker coefficients.
Let $\eta \in C^\infty(\Gamma \backslash \text{GL}_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\text{GL}_{n-1}(K)$.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Explanation for $\text{GL}_n$

Let $\eta \in C^\infty(\Gamma \backslash \text{GL}_n(A))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $\text{GL}_{n-1}(K)$.

Conjugate, restrict to the next column and continue

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
= \cdots
\]
Explanation for $GL_n$

Let $\eta \in C^\infty(\Gamma \setminus GL_n(\mathbb{A}))$. Restrict to the last column and decompose to Fourier series. All non-trivial characters are conjugate by $GL_{n-1}(\mathbb{K})$. Conjugate, restrict to the next column and continue:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} = \cdots
\]

\[
\begin{pmatrix}
0 & 0 & * & * & * & * \\
0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & * & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} = \cdots
\]
Example: $\text{Sp}(4)$

$$\text{sp}_4 = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : B = B^t \right\}. $$

Let $\mathfrak{n}$ be the Borel nilradical, and $\mathfrak{u} \subset \mathfrak{n}$ be the Siegel nilradical, spanned by $B$. Characters given by $\bar{\mathfrak{u}} \cong \text{Sym}^2(\mathbb{K}^2)$. Restricting $\eta$ to $B$ and decomposing into Fourier series we obtain $\eta = \sum_{f \in \bar{\mathfrak{u}}} \mathcal{F}_{\mathfrak{u}, f}[\eta]$.

1. **Constant term** $\mathcal{F}_{\mathfrak{u}, 0}[\eta]$: Restrict to the Siegel Levi $L \cong \text{GL}_2(\mathbb{A})$, and decompose to Fourier series on the abelian group $N \cap L$:

$$\mathcal{F}_{\mathfrak{u}, 0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a, 0}[\eta].$$
Example: \( \text{Sp}(4) \)

\[
\text{span}_4 = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\}.
\]

Let \( \mathfrak{n} \) be the Borel nilradical, and \( \mathfrak{u} \subset \mathfrak{n} \) be the Siegel nilradical, spanned by \( B \). Characters given by \( \bar{\mathfrak{u}} \cong \text{Sym}^2(\mathbb{K}^2) \). Restricting \( \eta \) to \( B \) and decomposing into Fourier series we obtain \( \eta = \sum_{f \in \bar{\mathfrak{u}}} \mathcal{F}_{\mathfrak{u},f}[\eta] \).

1. **Constant term** \( \mathcal{F}_{\mathfrak{u},0}[\eta] \): Restrict to the Siegel Levi \( L \cong \text{GL}_2(\mathbb{A}) \), and decompose to Fourier series on the abelian group \( N \cap L \):

\[
\mathcal{F}_{\mathfrak{u},0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta].
\]

2. Any \( f \) of rank one is conjugate under \( L \) to \( f_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

Decomposing \( \mathcal{F}_{\mathfrak{u},f_1}[\eta] \) on \( N \cap L \):

\[
\mathcal{F}_{\mathfrak{u},f_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,1}[\eta].
\]
\[ \text{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\}; \quad \eta = \sum_{f \in U} \mathcal{F}_{u,f}[\eta]. \]

1. Constant term \( \mathcal{F}_{u,0}[\eta] \): Decompose to Fourier series on \( N \cap L \):

\[ \mathcal{F}_{u,0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta]. \]
\( \mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B = B^t \\ C = C^t & -A^t \end{pmatrix} \right\}; \quad \eta = \sum_{f \in \mathcal{U}} \mathcal{F}_{u, f}[\eta]. \)

1. **Constant term \( \mathcal{F}_{u, 0}[\eta] \):** Decompose to Fourier series on \( N \cap L \):
   \[ \mathcal{F}_{u, 0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a, 0}[\eta]. \]

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Decomposing \( \mathcal{F}_{u, f_1}[\eta] \) on \( N \cap L \):
\[ \mathcal{F}_{u, f_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a, 1}[\eta]. \]
\[
\mathfrak{sp}_4 = \left\{ \begin{pmatrix} A & B \\ C = C^t & -A^t \end{pmatrix} \right\}; \quad \eta = \sum_{f \in \tilde{u}} \mathcal{F}_{u,f}[\eta].
\]

1. **Constant term** \( \mathcal{F}_{u,0}[\eta] \): Decompose to Fourier series on \( N \cap L \):

\[
\mathcal{F}_{u,0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta].
\]

2. Any \( f \) of rank one is conjugate under \( L \) to \( f_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

Decomposing \( \mathcal{F}_{u,f_1}[\eta] \) on \( N \cap L \):

\[
\mathcal{F}_{u,f_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,1}[\eta].
\]

3. **Split non-degenerate forms** are conjugate to \( f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Using Weyl group conjugation (24) and root exchange, we express \( \mathcal{F}_{u,f_2} \) through \( \mathcal{F}_{u',e_{21}} \), where \( u' = \text{Span}(e_{12} - e_{43}, e_{13}, e_{24}) \subset \mathfrak{n} \).

Fourier expansion by the remaining coordinate of \( e_{14} + e_{23} \in \mathfrak{n} \):

\[
\mathcal{F}_{u,f}[\eta](g) = \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta]((Id + xe_{24})wg).
\]
X:= set of anisotropic $2 \times 2$ forms. For $f \in X$, we cannot simplify $F_{u,f}[\eta]$. Summarizing, for any $\eta$ on $G = \text{Sp}_4(\mathbb{A})$ we have

$$
\eta(g) = \sum_{f \in X} F_{u,f}[\eta](g) + \sum_{a \in K} \left( \sum_{\gamma \in L/O(1,1)} \int_{x \in \mathbb{A}} W_{1,a}[\eta](v_x w \gamma g) + \sum_{\gamma \in L/(N \cap L)} W_{a,1}[\eta](\gamma g) + W_{a,0}[\eta](g) \right)
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If $\eta$ is cuspidal then $\mathcal{W}_{0,a}[\eta] = \mathcal{W}_{a,0}[\eta] = 0$. If $\eta$ is non-generic, then $\mathcal{W}_{1,a}[\eta] = \mathcal{W}_{a,1}[\eta] = 0$, unless $a = 0$. Thus

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3. If $\eta$ is cuspidal and non-generic then $\eta = \sum_{f \in X} \mathcal{F}_{u,f}[\eta]$. 
Parabolic minimal Fourier coeff. of next-to-minimal forms

- $g$ split simply laced, $\mathfrak{h} \subset g$ Cartan, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ Borel
Parabolic minimal Fourier coeff. of next-to-minimal forms

- $g$ split simply laced, $\mathfrak{h} \subset g$ Cartan, $\mathfrak{b} = \mathfrak{h} \oplus u$ Borel
- $\alpha$ simple root. $q_\alpha = l_\alpha \oplus n_\alpha = g_{\geq 0}$ max. parabolic.
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- $I(\perp \alpha) = \{\beta_1, \ldots, \beta_k\}$ Bourbaki enumeration of the simple roots orthogonal to $\alpha$. 

Theorem

$F_{\alpha}, f[\eta_{ntm}] = \sum_{\gamma \in \Gamma} \sum_{\phi \in \mathfrak{g} \times -\beta_i} W_\phi + f[\eta_{ntm}](\gamma g)$
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Dmitry Gourevitch

Fourier coefficients

February 2021 21 / 24
Parabolic minimal Fourier coeff. of next-to-minimal forms

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- $L_i \supset S_i :=$ stabilizer of the root space $g_{-\beta_i}$, $\Gamma_i := (L_i \cap \Gamma)/(S_i \cap \Gamma)$.
- For $f \in g^{\times}_{-\alpha}$ and next-to-minimal $\eta_{ntm} \in C^\infty(\Gamma \setminus G)$ let

$$A^f_i[\eta_{ntm}](g) := \sum_{\gamma \in \Gamma_{i-1}} \sum_{\varphi \in g^\times_{-\beta_i}} \mathcal{W}_{\varphi + f}[\eta_{ntm}](\gamma g)$$
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\[ A^f_i[\eta_{ntm}](g) := \sum_{\gamma \in \Gamma_{i-1}} \sum_{\varphi \in \mathfrak{g}_{-\beta_i}^\times} \mathcal{N}_{\varphi+f}[\eta_{ntm}](\gamma g) \]
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- $\forall\ i \ G \supset L_i := \text{Levi given by roots } \beta_1, \ldots, \beta_i$
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**Theorem**

$$\mathcal{F}_{S_{\alpha}, f}[\eta_{ntm}] = \mathcal{W}_f[\eta_{ntm}] + \sum_{i=1}^{k} A_i^f[\eta_{ntm}]$$
\( g \) split simply laced, \( h \subset g \) Cartan, \( b = h \oplus u \) Borel
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\begin{itemize}
  \item $g$ split simply laced, $\mathfrak{h} \subset g$ Cartan, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ Borel
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    \[
    A_i^f[\eta_{ntm}](g) := \sum_{\gamma \in \Gamma_{i-1}} \sum_{\varphi \in g_{-\beta_i}^\times} \mathcal{W}_{\varphi + f}[\eta_{ntm}](\gamma g)
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```
  2
 /|
/  |
2---3---4---5---6---n-1---n
```


\( F := \text{local field of char. } 0, \ G := \mathbf{G}(F), \ g := \text{Lie}(G), \)
- $F := \text{local field of char. 0, } G := G(F), \ g := \text{Lie}(G)$,
- $(H, f) \in g \times g \text{ Whittaker pair } u := g_{\geq 1}^H, \ n_{H,f} := (g_1^H \cap g^f) \oplus g_{\geq 1}^H$. 

All the theorems above have local analogues with similar proofs.
\( F \) := local field of char. 0, \( G := G(F) \), \( \mathfrak{g} := \text{Lie}(G) \),

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\( u/\mathfrak{n}_{H,f} \) is a symplectic space, and its Heisenberg group \( \mathcal{H} \) is a quotient of \( U \).
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- $\forall$ smooth representation $\pi$, define its $(H, f)$-Whittaker quotient by

$$\pi_{H, f} := \mathcal{W}_{H, f} \otimes_G \pi \simeq \pi_{I, \chi}.$$
Local picture

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- All the theorems above have local analogues with similar proofs.
Wave front set and wave-front cycle

Let $\pi$ be smooth, admissible and finitely generated.

**Theorem (Howe, Harish-Chandra, Barbasch-Vogan 70s)**

Near $e \in G$, the character distribution equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

$$\exp^* (\chi_\pi) \approx \sum c_\mathcal{O} \mathcal{F} (\mu_\mathcal{O})$$
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- Let \( \mathcal{N} \subset g \) denote the nilpotent cone.
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\]

- Let \( \mathcal{N} \subset g \) denote the nilpotent cone.
- \( \text{WF}(\pi) := \bigcup \{ \overline{O} \mid c_O \neq 0 \} \subset \mathcal{N} \).
- \( \text{WF}^{\text{max}}(\pi) := \text{union of maximal orbits in } \text{WF}(\pi) \).

**Theorem (Moeglin-Waldspurger, 87')**

Let \( F \) be \( p \)-adic and let \((H, f)\) be a Whittaker pair.

- If \( \pi_{H,f} \neq 0 \) then \( f \in \text{WF}(\pi) \).
Wave front set and wave-front cycle

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Theorem (Moeglin-Waldspurger, 87’)

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