Degenerate Whittaker functionals for real reductive groups

Dmitry Gourevitch & Siddhartha Sahi
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Whittaker functionals

- $\mathbb{F}$: local field of char.0, $G$: reductive group over $\mathbb{F}$. 

Theorem (Gelfand-Kazhdan, Shalika)

For $\pi \in \text{Irr}(G)$, $\psi \in \Psi \times$, $\dim \text{Wh}^*\psi(\pi) \leq 1$. 

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- $\mathbb{F}$: local field of char.0, $G$: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible representations (of moderate growth).

Assume $G$ quasisplit: fix Borel $B = HN$, $n = \text{Lie } F(N)$. Define $n' = [n, n]$, $v = n/n'$, $\Psi = v^* \subset n^*$, $\Psi \leftrightarrow \text{Lie algebra characters of } n \leftrightarrow \text{unitary group characters of } N$. $\Psi \supset \Psi \times = \text{non-degenerate characters.}$

For $\psi \in \Psi \times$, $\pi \in \mathcal{M}(G)$ define

$\text{Wh}^* \psi(\pi) = \text{Hom}_{cts} N(\pi, \psi), \Psi(\pi) = \{ \psi \in \Psi: \text{Wh}^* \psi(\pi) \neq 0 \}$

(Casselman) For any $\psi \in \Psi \times$, $\pi \mapsto \text{Wh}^* \psi(\pi)$ is an exact functor.

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Kostant’s theorem

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Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

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- \( \pi \) is called large if \( \text{WF}(\pi) = \mathcal{N} \).
In the p-adic case, Moeglin and Waldspurger give a very general definition of degenerate Whittaker models and give a precise connection between their existence and the wave-front set $WF(\pi)$. In the real case there is no full analog currently.
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We consider degenerate functionals $\sim$ arbitrary characters of $n$. 
Main results

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Theorem (1)

For \( \pi \in \mathcal{M}(G) \) we have
\[
\Psi(\pi) \subset WF(\pi) \subset \Psi(\tilde{\pi})
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Moreover if \( G = \text{GL}_n(\mathbb{R}) \) or if \( G \) is a complex group then
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\tilde{\pi} = \pi \quad \text{and} \quad \Psi(\pi) = WF(\pi) \cap \Psi(\tilde{\pi})
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Main results

Theorem (2)

The sets $\Psi(\pi)$ and $WF(\pi)$ determine one another if

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Key observation for the second statement:

Theorem (3)

Let $\mathcal{O}$ be a nilpotent orbit for a complex classical Lie algebra then $\mathcal{O}$ is uniquely determined by $\overline{\mathcal{O}} \cap \Psi$. 

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Uniqueness

- An analog of Shalika’s result would be uniqueness of "minimally degenerate" Whittaker models. So far it is known only for $GL_n$, both in real and p-adic cases.

- Let $G$ be $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$. For a partition $\lambda$ of $n$ let $O_\lambda$ denote the corresponding nilpotent orbit and $\psi_\lambda$ denote the corresponding character of $N$. 

Theorem (Aizenbud-G-Sahi) Let $\pi \in \hat{G}$ be an irreducible unitary representation. Let $\lambda$ be such that $WF(\pi) = O_\lambda$. Then $\dim Wh(\psi_\lambda)(\pi) = 1$.

In the p-adic case the analogous theorem was proven by Zelevinsky without the assumption that $\pi$ is unitary. The proofs in both cases use "derivatives".
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Algebraic setting

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- Let \( K \subset G \) be maximal compact subgroup. A \((\mathfrak{g}, K)\)-module is a complex vector space with compatible actions of \( \mathfrak{g} \) and \( K \) such that every vector is \( K \)-finite.
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- Let \( \mathcal{HC}(G) \) denote the category of \((g, K)\)-modules of finite length.

Theorem (Casselman-Wallach)

The functor \( \pi \mapsto \pi_{\mathcal{K} - \text{finite}} \) is an equivalence of categories \( M(G) \cong \mathcal{HC}(G) \).

For \( M \in \mathcal{HC}(G) \) and \( \psi \in \Psi \), we define \( Wh'_{\psi}(M) := \text{Hom}(M, \psi) \), \( \Psi(M) := \{ \psi \in \Psi | Wh'_{\psi}(M) \neq 0 \} \).

Kostant showed that \( \pi \) is generic iff \( \pi_{\mathcal{K} - \text{finite}} \) is generic, though dimensions of Whittaker spaces differ considerably.
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Schmid and Vilonen proved that \(\text{WF}(\pi)\) and \(\text{As}_V(\pi^{K-\text{finite}})\) determine each other.
Associated varieties and our algebraic theorem

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- Let $\text{pr}_n^* : \mathfrak{g}^* \to \mathfrak{n}^*$ denote the natural projection (restriction to $\mathfrak{n}$).
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Let \( \text{pr}_{n^*} : g^* \to n^* \) denote the natural projection (restriction to \( n \)).

**Theorem (0)**

For \( M \in \mathcal{HC} \) we have \( \Psi(M) = \text{pr}_{n^*}(\text{As}\mathcal{V}(M)) \cap \Psi \).
Idea of the proof

Since $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is commutative, from Nakayama’s lemma we have $\Psi(M) = \text{Supp}(M/[\mathfrak{n}, \mathfrak{n}]M)$. Now, restriction to $\mathfrak{n}$ corresponds to projection on $\mathfrak{n}^*$ and quotient by $[\mathfrak{n}, \mathfrak{n}]$ corresponds to intersection with $\Psi = [\mathfrak{n}, \mathfrak{n}]^\perp$. 

However, in non-commutative situation one could even have $\mathcal{V} = [\mathfrak{n}, \mathfrak{n}]\mathcal{V}$. For example, let $G = \text{GL}(3, \mathbb{R})$ and consider the identification of $\mathfrak{n}$ with the Heisenberg Lie algebra $\langle x, dx, 1 \rangle$ acting on $\mathcal{V} = \mathbb{C}[x]$. Let $\mathfrak{b} = h + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$, let $\mathcal{V}$ be a $\mathfrak{b}$-module. We define the $\mathfrak{n}$-adic completion and Jacquet module as follows: $\hat{\mathcal{V}} = \hat{\mathcal{V}}^\mathfrak{n} = \lim_{\leftarrow} \mathcal{V}/n_i \mathcal{V}$, $J(\mathcal{V}) = J_\mathfrak{b}(\mathcal{V}) = (\hat{\mathcal{V}}^\mathfrak{n})_h$-finite.

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Let $b = h + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$, let $V$ be a $b$-module. We define the $\mathfrak{n}$-adic completion and Jacquet module as follows: $\hat{V} = \hat{V}_{\mathfrak{n}} = \lim_{\leftarrow} V/\mathfrak{n}^iV$, $J(V) = J_b(V) = (\hat{V}_{\mathfrak{n}})_h$-finite. 

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$$\hat{V} = \hat{V}_n = \lim_{\leftarrow} V/n^i V, \quad J(V) = J_{\mathfrak{b}}(V) = \left(\hat{V}_n\right)_{\mathfrak{h}}\text{-finite}$$
Sketch of the proof

- Define $n' = [n, n]$ and $CV = H_0(n', V) = V / n' V$. 

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Sketch of the proof

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(Easy) \( J(CM) \approx C(JM) \) as \( \mathfrak{b} \)-modules.
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- (Easy) \( J(CM) \approx C(JM) \) as \( \mathfrak{b} \)-modules.
- (Bernstein+Joseph)
  \( \text{An} \mathcal{V}_v(J(CM)) = \text{As} \mathcal{V}_v(C(JM)) = \text{As} \mathcal{V}_n(JM) \cap \Psi. \)
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(Easy) $J(CM) \approx C(JM)$ as $\mathfrak{b}$-modules.

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Define $n' = [n, n]$ and $CV = H_0(n', V) = V / n'V$.

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(Casselman-Osborne+Gabber) $\text{As}\mathcal{V}_n(M) = pr_n^* (\text{As}\mathcal{V}_g(M))$. 
Sketch of the proof

- Define \( n' = [n, n] \) and \( CV = H_0(n', V) = V / n' V \).
- (Nakayama) \( \Psi(M) = \text{Supp}_v(CM) = \text{An}_v(CM) \)
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- (Ginzburg+ENV) \( \text{As}_n(JM) \supset \text{As}_n(M) \cap \Psi. \)
- (Casselman-Osborne+Gabber) \( \text{As}_n(M) = pr_n^*(\text{As}_g(M)). \)
- Thus \( \Psi(M) \supset pr_n^*(\text{As}_g(M)) \cap \Psi; \) other inclusion is easy.
Proof of Theorem 3

Proof.

For $GL(n, \mathbb{R})$ and $SL(n, \mathbb{C}) \sim$ Jordan form

- Orbits for $Sp_{2n}(\mathbb{C})$ or $O_n(\mathbb{C}) \sim$ partitions satisfying certain conditions
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- For each partition $\lambda$ and each $k$ there is a partition $\mu \leq \lambda$, which meets $\Psi$ and satisfies $\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$

Result for $SO_n(\mathbb{C})$ requires slight additional argument.
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- Result for $\text{SO}_n(\mathbb{C})$ requires slight additional argument.
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Degenerate Whittaker functionals

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  2. $F_4(a_3)$ and $C_3(a_1)$
- For $G = E_6$:
  1. $E_6(a_1)$ and $D_5$
  2. $D_4(a_1)$ and $A_3 + A_1$
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Counterexamples for exceptional groups

For $G = E_8$:
Counterexamples for exceptional groups

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6. $E_8(a_7)$, $E_7(a_5)$, $E_6(a_3) + A_1$, and $D_6(a_2)$. 