Whittaker functionals

- $F$: local field of char.0, $G$: reductive group over $\mathbb{F}$.
- $\mathcal{M}(G)$: smooth admissible representations (of moderate growth).
- Assume $G$ quasisplit: fix Borel $B = HN$, $n := \text{Lie}_F(N)$.

Define $n' = [n, n], v = n/n', \Psi = v^* \subset n^*$

- $\Psi \leftrightarrow$ Lie algebra characters of $n \leftrightarrow$ unitary group characters of $N$
- $\Psi \supset \Psi^\times :=$ non-degenerate characters.
- For $\psi \in \Psi, \pi \in \mathcal{M}(G)$ define

  $$Wh^*_\psi(\pi) := \text{Hom}_{ct}^N(\pi, \psi), \Psi(\pi) := \{\psi \in \Psi : Wh^*_\psi(\pi) \neq 0\}$$

- (Casselman) For any $\psi \in \Psi^\times$, $\pi \mapsto Wh^*_\psi(\pi)$ is an exact functor.

**Theorem (Gelfand-Kazhdan, Shalika)**

For $\pi \in \text{Irr}(G), \psi \in \Psi^\times, \dim Wh^*_\psi(\pi) \leq 1.$
Kostant’s theorem

We say $\pi$ is \textit{generic} if $\exists \psi \in \Psi^\times$ s.t. $Wh_\psi(\pi) \neq 0$.

\textbf{Theorem (Kostant, Rodier)}

$\pi$ is generic iff it is large.

\textbf{Theorem (Harish-Chandra, Howe)}

Near $e \in G$, the character distribution (asymptotically) equals to a linear combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits.

\[ \chi_\pi \approx \sum a_\mathcal{O} \mathcal{F}(\mu_\mathcal{O}) \]

- Define $WF(\pi) = \cup \{ \mathcal{O} | a_\mathcal{O} \neq 0 \} \subset \mathcal{N}$, where $\mathcal{N} \subset g^*$ denotes the nilpotent cone.
- $\pi$ is called \textit{large} if $WF(\pi) = \mathcal{N}$. 

In the p-adic case, Moeglin and Waldspurger give a very general definition of degenerate Whittaker models and give a precise connection between their existence and the wave-front set $WF(\pi)$. In the real case there is no full analog currently.

Several authors (Matumoto, Yamashita, ... ) consider generalized Whittaker functionals $\sim$ generic characters for smaller nilradicals.

We consider degenerate functionals $\sim$ arbitrary characters of $\mathfrak{n}$. 
Main results

- From now on, let $\mathbb{F} = \mathbb{R}$.
- The finite group $F_G = \text{Norm}_{G_C}(G) / (Z_{G_C} \cdot G)$ acts on $\mathcal{M}(G)$.
- For $\pi \in \mathcal{M}(G)$, define $\tilde{\pi} = \bigoplus \{ \pi^a : a \in F_G \}$.

**Theorem (1)**

For $\pi \in \mathcal{M}(G)$ we have

$$\Psi(\pi) \subset \text{WF}(\pi) \cap \Psi \subset \Psi(\tilde{\pi}) \quad (1)$$

Moreover if $G = GL_n(\mathbb{R})$ or if $G$ is a complex group then $\tilde{\pi} = \pi$ and

$$\Psi(\pi) = \text{WF}(\pi) \cap \Psi \quad (2)$$
Main results

Theorem (2)

The sets $\Psi(\pi)$ and $WF(\pi)$ determine one another if

1. $G = GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ or $SL_n(\mathbb{C})$ and $\pi \in \mathcal{M}(G)$
2. $G = Sp_{2n}(\mathbb{C})$ or $O_n(\mathbb{C})$ or $SO_n(\mathbb{C})$ and $\pi$ is irreducible

Key observation for the second statement:

Theorem (3)

Let $\mathcal{O}$ be a nilpotent orbit for a complex classical Lie algebra then $\mathcal{O}$ is uniquely determined by $\overline{\mathcal{O}} \cap \Psi$. 
An analog of Shalika’s result would be uniqueness of “minimally degenerate” Whittaker models. So far it is known only for \( GL_n \), both in real and p-adic cases.

Let \( G \) be \( GL_n(\mathbb{R}) \) or \( GL_n(\mathbb{C}) \). For a partition \( \lambda \) of \( n \) let \( O_\lambda \) denote the corresponding nilpotent orbit and \( \psi_\lambda \) denote the corresponding character of \( N \).

**Theorem (Aizenbud-G-Sahi)**

Let \( \pi \in \hat{G} \) be an irreducible unitary representation. Let \( \lambda \) be such that \( WF(\pi) = \overline{O_\lambda} \). Then \( \dim Wh_{\psi_\lambda}(\pi) = 1 \).

In the \( p \)-adic case the analogous theorem was proven by Zelevinsky without the assumption that \( \pi \) is unitary. The proofs in both cases use ”derivatives”. 
Algebraic setting

- From now on, we let $\mathfrak{n}, \mathfrak{g}$, etc. denote complexified Lie algebras.
- Let $K \subset G$ be maximal compact subgroup. A $(\mathfrak{g}, K)$-module is a complex vector space with compatible actions of $\mathfrak{g}$ and $K$ such that every vector is $K$-finite.
- Let $\mathcal{HC}(G)$ denote the category of $(\mathfrak{g}, K)$-modules of finite length.

**Theorem (Casselman-Wallach)**

The functor $\pi \mapsto \pi^K_{\text{finite}}$ is an equivalence of categories $\mathcal{M}(G) \cong \mathcal{HC}(G)$

- For $M \in \mathcal{HC}(G)$ and $\psi \in \Psi_C$ we define
  
  $$\text{Wh}_{\psi}'(M) := \text{Hom}_n(M, \psi), \quad \Psi(M) := \{ \psi \in \Psi_C \mid \text{Wh}_{\psi}'(M) \neq 0 \}$$

- Kostant showed that $\pi$ is generic iff $\pi^K_{\text{finite}}$ is generic, though dimensions of Whittaker spaces differ considerably.
Using PBW filtration, $\text{gr}\, \mathcal{U}(g) = \text{Sym}(g) = \text{Pol}(g^*)$.

Using this, one can define

$$\text{As}\mathcal{V}(M) \subset \text{An}\mathcal{V}(M) \subset \mathcal{N}$$

Schmid and Vilonen proved that $\text{WF}(\pi)$ and $\text{As}\mathcal{V}(\pi^{K-\text{finite}})$ determine each other.

Let $pr_{n^*} : g^* \to n^*$ denote the natural projection (restriction to $n$).

**Theorem (0)**

For $M \in \mathcal{HC}$ we have $\Psi(M) = pr_{n^*}(\text{As}\mathcal{V}(M)) \cap \Psi$. 
Idea of the proof

- Since $n/[n, n]$ is commutative, from Nakayama’s lemma we have $\Psi(M) = \text{Supp}(M/[n, n]M)$. Now, restriction to $n$ corresponds to projection on $n^*$ and quotient by $[n, n]$ corresponds to intersection with $\Psi = [n, n]^\perp$.

- However, in non-commutative situation one could even have $V = [n, n]V$. For example, let $G = GL(3, \mathbb{R})$ and consider the identification of $n$ with the Heisenberg Lie algebra $\langle x, \frac{d}{dx}, 1 \rangle$ acting on $V = \mathbb{C}[x]$.

- Let $\mathfrak{b} = \mathfrak{h} + n$ be the Borel subalgebra of $\mathfrak{g}$, let $V$ be a $\mathfrak{b}$-module. We define the $n$-adic completion and Jacquet module as follows:
  
  $\hat{V} = \hat{V}_n = \lim_{\leftarrow} V/n^i V, \quad J(V) = J_\mathfrak{b}(V) = \left(\hat{V}_n\right)^{\mathfrak{h}\text{-finite}}$
Sketch of the proof

- Define \( n' = [n, n] \) and \( CV = H_0(n', V) = V / n'V \).
- (Nakayama) \( \Psi(M) = \text{Supp}_v(CM) = \text{An}_v(CM) \)
- (Joseph+Gabber) \( \text{An}_v(CM) = \text{An}_v(C\widehat{M}) = \text{An}_v(J(CM)) \)
- (Easy) \( J(CM) \approx C(JM) \) as \( b \)-modules.
- (Bernstein+Joseph) \( \text{An}_v(J(CM)) = \text{As}_v(CJM) = \text{As}_n(JM) \cap \Psi \).
- (Ginzburg+ENV) \( \text{As}_n(JM) \supset \text{As}_n(M) \cap \Psi \).
- (Casselman-Osborne+Gabber) \( \text{As}_n(M) = pr_n^*(\text{As}_g(M)) \).
- Thus \( \Psi(M) \supset pr_n^*(\text{As}_g(M)) \cap \Psi \); other inclusion is easy.
Proof of Theorem 3

Proof.

For $\text{GL}(n, \mathbb{R})$ and $\text{SL}(n, \mathbb{C}) \sim$ Jordan form

- Orbits for $\text{Sp}_{2n}(\mathbb{C})$ or $\text{O}_n(\mathbb{C}) \sim$ partitions satisfying certain conditions
- An orbit meets $\Psi$ iff it has at most one part $\geq 2$ with odd multiplicity
- For each partition $\lambda$ and each $k$ there is a partition $\mu \leq \lambda$, which meets $\Psi$ and satisfies $\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$
- Result for $\text{SO}_n(\mathbb{C})$ requires slight additional argument.
Counterexamples for exceptional groups

Fact

*Theorem 3 is false for every exceptional group.*

- We list all orbits whose closures have the same intersection with $\Psi$.
- We follow Bala-Carter notation and we have underlined the special orbits.
- For $G = G_2$: $G_2(a_1)$ and $\widetilde{A}_1$
- For $G = F_4$:
  1. $F_4(a_1)$ and $F_4(a_2)$
  2. $F_4(a_3)$ and $C_3(a_1)$
- For $G = E_6$:
  1. $E_6(a_1)$ and $D_5$
  2. $D_4(a_1)$ and $A_3 + A_1$
- For $G = E_7$:
  1. $E_7(a_1)$ and $E_7(a_2)$
  2. $E_7(a_3)$ and $D_6$
  3. $E_6(a_1)$ and $E_7(a_4)$. 
Counterexamples for exceptional groups

For $G = E_8$:

1. $E_8(a_1), E_8(a_2),$ and $E_8(a_3)$
2. $E_8(a_4), E_8(b_4)$ and $E_8(a_5)$
3. $E_7(a_1), E_8(b_5)$ and $E_7(a_2)$
4. $E_8(a_6)$ and $D_7(a_1)$
5. $E_6(a_1)$ and $E_7(a_4)$
6. $E_8(a_7), E_7(a_5), E_6(a_3) + A_1,$ and $D_6(a_2)$. 