EXERCISE 3 IN D-MODULES I

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(1) Exterior products of $D$-modules. Consider to affine spaces $X,Y$. Let $M$ be a $D(X)$-module and $N$ be a $D(Y)$-module.

(a) Consider the vector space $M \otimes_k N$ and define on it a structure of a $D(X \times Y)$-module. This module is called the exterior product of $M$ and $N$ and denoted $M \otimes N$.

(b) Prove that $d(M \otimes N) = d(M) + d(N)$ and $e(M \otimes N) = e(M)e(N)$.

(2) Tensor product over $O$. Let $M,N$ be $D(X)$-modules (where $X$ is an affine space). Consider the space $M \otimes N := M \otimes_{O(X)} N$ and define the structure of a $D(X)$-module by Leibnitz rule.

(a) Show that $M \otimes N$ is canonically isomorphic to the module $\Delta^0(M \square N)$, where $\Delta : X \to X \times X$ is the diagonal embedding

(b) Interpret the analytic meaning of this algebraic operation. Show that the inner tensor product $M \otimes N$ of finitely-generated $D$-modules is not always finitely generated.

(3) (P) Show that if $M,N$ are holonomic then $M \otimes N$ is also holonomic.

(4) (P) Let $M$ be a left $D(X)$-module and $N$ be a right $D(X)$-module. Define a natural structure of a right $D(X)$-module on $M \otimes N$. Explain the analytic meaning of this construction. Convince yourself that there is no natural tensor product of right $D(X)$-modules.

(5) (P) Let $X$ be a real vector space. Fix a real polynomial $P$ on $X$. Fix a not very singular distribution $\xi$ on $X$ (e.g. a finite sum of differential operators applied to continuous functions) and define a family of generalized functions $G(\lambda) := P^\lambda f$ for $\text{Re} \lambda >> 0$. Show that if $\xi$ is holonomic then the family extends meromorphically to the whole complex plane. More precisely, show that there exists a differential operator $d \in D(X)[\lambda]$ and a polynomial $b \in C[\lambda]$ such that $d(\lambda)G(\lambda + 1) = b(\lambda)G(\lambda)$.

(6) (P)

(a) Similarly to the previous problem, show that the function $G(\lambda, \mu) = P^\lambda G_\mu \xi$ has a meromorphic continuation in two variables $\lambda, \mu$.

(b) Show that the function $b(\lambda, \mu)$ can be chosen to be a product of linear functions.

(7) (P) Let $T$ be a differential operator with constant coefficients on $\mathbb{R}^n$. Show that there exists a distribution $f$ with $Tf = \delta_0$. Moreover, $f$ can be chosen to be a tempered holonomic distribution.

(8) (P) Let $M$ be a $D_n$-module generated by a finite subset $S$. Let $I \subset D_n$ be the annihilator of $S$, and let $J \subset k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ be the ideal generated by the symbols of the elements of $I$. Show that the associated variety $AV(M)$ is the zero set of $J$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/dmod1.html

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