EXERCISE 3 IN D-MODULES I

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(1) Exterior products of $\mathcal{D}$-modules. Consider to affine spaces $X,Y$. Let $M$ be a $\mathcal{D}(X)$-module and $N$ be a $\mathcal{D}(Y)$-module.

(a) Consider the vector space $M \otimes_k N$ and define on it a structure of a $\mathcal{D}(X \times Y)$-module. This module is called the exterior product of $M$ and $N$ and denoted $M \boxtimes N$.

(b) Prove that $d(M \boxtimes N) = d(M) + d(N)$ and $e(M \boxtimes N) \leq (d(M) + d(N))e(M)e(N)$.

(2) Tensor product over $\mathcal{O}$. Let $M,N$ be $\mathcal{D}(X)$-modules (where $X$ is an affine space). Consider the space $M \otimes \mathcal{O}(X) N$ and define the structure of a $\mathcal{D}(X)$-module by Leibnitz rule.

(a) Show that $M \otimes \mathcal{O}(X) N$ is canonically isomorphic to the module $\Delta^0(M \boxtimes N)$, where $\Delta : X \rightarrow X \times X$ is the diagonal embedding.

(b) Interpret the analytic meaning of this algebraic operation. Show that the inner tensor product $M \otimes N$ of finitely-generated $\mathcal{D}$-modules is not always finitely generated.

(3) (P) Show that if $M,N$ are holonomic then $M \otimes N$ is also holonomic.

(4) (P) Let $M$ be a left $\mathcal{D}(X)$-module and $N$ be a right $\mathcal{D}(X)$-module. Define a natural structure of a right $\mathcal{D}(X)$-module on $M \otimes N$. Explain the analytic meaning of this construction. Convince yourself that there is no natural tensor product of right $\mathcal{D}(X)$-modules.

(5) (P) Let $X$ be a real vector space. Fix a real polynomial $P$ on $X$. Fix a not very singular distribution $\xi$ on $X$ (e.g. a finite sum of differential operators applied to continuous functions) and define a family of generalized functions $G(\lambda) := P^{\lambda} \xi$ for $\text{Re} \lambda >> 0$. Show that if $\xi$ is holonomic then the family extends meromorphically to the whole complex plane. More precisely, show that there exists a differential operator $d \in \mathcal{D}(X)[\lambda]$ and a polynomial $b \in \mathbb{C}[\lambda]$ such that $d(\lambda)G(\lambda + 1) = b(\lambda)G(\lambda)$.

(6) (P)

(a) Similarly to the previous problem, show that the function $G(\lambda, \mu) = P^{\lambda}G^{\mu} \xi$ has a meromorphic continuation in two variables $\lambda, \mu$.

(b) (*)Show that the function $b(\lambda, \mu)$ can be chosen to be a product of linear functions.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html