EXERCISE 4 IN D-MODULES I

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Let $A$ be a Noetherian algebra, $\mathcal{M}(A)$ denote the category of $A$-modules and $\mathcal{M}^f(A)$ denote the subcategory of finitely-generated $A$-modules.

(1) Show that $P \in \mathcal{M}^f(A)$ is projective in $\mathcal{M}^f(A)$ if and only if it is projective in $\mathcal{M}(A)$. Show that the homological dimensions of $\mathcal{M}^f(A)$ and $\mathcal{M}(A)$ are equal.

(2) Show that a module $M$ is finitely-generated if and only if for any system of submodules satisfying $\sum M_\alpha = M$ there exists a finite subsystem with this property.

(3) (P) Let $M \in \mathcal{M}^f(A)$. Suppose that $M$ has a projective resolution of length $d$. Consider the functor $E : N \mapsto \text{Ext}^d(M,N)$. Show that there exists a right $A$-module $R$ such that this functor is isomorphic to a functor $T_R$ defined by $T_R(N) := R \otimes_A N$. Show that the module $R$ is defined uniquely up to canonical isomorphism.

(4) (P) Let $\mathcal{C}$ be an abelian category. Let $\Pi \in \mathcal{C}$ be a projective object. Suppose that arbitrary direct powers of $\Pi$ are defined, and for any object $M \in \mathcal{C}$ there exist a power of $\Pi$ and an epimorphism $\Pi^\alpha \twoheadrightarrow M$. Show that the $\mathcal{C}$ is equivalent to the category of right modules over the ring $\text{End}(\Pi)$.

Direct limits.

Definition 1. Let $I$ be a partially ordered set. We will consider it as a category with one morphism $i \to j$ if $i \leq j$, and no morphisms otherwise. An $I$-system of objects in a category $\mathcal{M}$ is a functor $I \to \mathcal{M}$. $I$ is called directed if for any $i,j \in I$ there exists $l \in I$ with $i,j \leq l$. The direct limit (or a colimit) $\varinjlim F$ of a system $F : I \to \mathcal{M}$ is an object $A \in \mathcal{M}$ and an isomorphism of the functors $\text{Hom}(A,\cdot)$ and the functor $G$ that sends every object $B \in \mathcal{M}$ to the set of natural transformations between $F$ and the constant functor $I \to \mathcal{M}$ that sends every object to $B$ and every map to identity. Sometimes $\varinjlim F$ denotes just the object $A$.

(5) Construct colimits in the category of sets and in $\mathcal{M}(A)$.

(6) Show that any $M \in \mathcal{M}(A)$ is a direct limit of a directed system in $\mathcal{M}^f(A)$.

(7) Show that if $I$ is a directed system and $\mathcal{M}$ an abelian category then the functor $F \mapsto \varinjlim F$ is exact.

(8) (P) Show that an $A$-module $M$ is finitely-generated if and only if the functor $\mathcal{M}(A) \to Ab$ given by $N \mapsto \text{Hom}(M,N)$ commutes with arbitrary directed direct limits. Moreover, show that if $M \in \mathcal{M}^f(A)$ then $\text{Ext}^i(M,\cdot)$ commutes with directed direct limits, and $\text{Hom}(M,\cdot)$ commutes with arbitrary direct limits. Do $\text{Ext}^i(M,\cdot)$ commute with arbitrary direct limits?

URL: http://www.wisdom.weizmann.ac.il/~dimagur/Dmod1.html

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