EXERCISE 5 IN D-MODULES I

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(1) (P) Define Φ : \( D_1 \oplus D_1 \to D_1 \) by \( \Phi(a, b) := a\partial + bx \). Show that Φ is onto and Ker Φ is isomorphic to the ideal \( D_1(x^2, \partial x) \). Conclude that this ideal is a projective module.

(2) Let \( V \) be a vector space, and \( a_1, \ldots, a_n : V \to V \) be a regular sequence of commuting linear operators. Then the Koszul complex is acyclic outside 0, and

\[ H_0(C) \simeq V / (a_1V + \cdots + a_nV). \]

(See the lecture for the definitions of regular sequence and Koszul complex)

(3) (*) \( \text{hd}(\mathcal{M}(R)) = \text{hd}(R) \).

(4) Let \( R \) be a ring and \( M \in \mathcal{M}^f(R) \) with a good filtration. Then

(i) for some \( l \) there exists a good filtration on \( R^l \) and a strict epimorphism \( R^l \to M \).

(ii) If \( \text{Gr} M \) is free then \( M \) is free.

(5) (P) Let \( L := k[x, x^{-1}], M := k[x] \) and \( C := L/M \). Note that they are all holonomic and consider the exact sequence

\[ 0 \to M \to L \to C \to 0. \]

Compute the dual D-modules, and describe the dual exact sequence

\[ 0 \to D(C) \to D(L) \to D(M) \to 0 \]

in terms of distributions.

(6) Let \( \mathcal{C} \) be an abelian category. Let \( \Pi \in \mathcal{C} \) be a projective object. Suppose that arbitrary direct powers of \( \Pi \) are defined, and for any object \( M \in \mathcal{C} \) there exist a power of \( \Pi \) and an epimorphism \( \Pi^a \to M \). Show that the \( \mathcal{C} \) is equivalent to the category of right modules over the ring \( \text{End}(\Pi) \).

Direct limits.

**Definition 1.** Let \( I \) be a partially ordered set. We will consider it as a category with one morphism \( i \to j \) if \( i \leq j \), and no morphisms otherwise. An \( I \)-system of objects in a category \( \mathcal{M} \) is a functor \( I \to \mathcal{M} \). \( I \) is called directed if for any \( i, j \in I \) there exists \( l \in I \) with \( i, j \leq l \). The direct limit (or a colimit) \( \lim_{\rightarrow} F \) of a system \( F : I \to \mathcal{M} \) is an object \( A \in \mathcal{M} \) and an isomorphism of the functors \( \text{Hom}(A, \cdot) \) and the functor \( G \) that sends every object \( B \in \mathcal{M} \) to the set of natural transformations between \( F \) and the constant functor \( I \to \mathcal{M} \) that sends every object to \( B \) and every map to identity. Sometimes \( \lim_{\rightarrow} F \) denotes just the object \( A \).

Let \( A \) be a Noetherian algebra, \( \mathcal{M}(A) \) denote the category of \( A \)-modules and \( \mathcal{M}^f(A) \) denote the subcategory of finitely-generated \( A \)-modules.

(3) Show that a module \( M \) is finitely-generated if and only if for any system of submodules satisfying \( \sum M_i = M \) there exists a finite subsystem with this property.

(4) Construct colimits in the category of sets and in \( \mathcal{M}(A) \).

(5) Show that any \( M \in \mathcal{M}(A) \) is a direct limit of a directed system in \( \mathcal{M}^f(A) \).

(6) Show that if \( I \) is a directed system and \( \mathcal{M} \) an abelian category then the functor \( F \mapsto \lim_{\rightarrow} F \) is exact.

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(7) Show that an $A$-module $M$ is finitely-generated if and only if the functor $\mathcal{M}(A) \to Ab$ given by $N \mapsto \text{Hom}(M,N)$ commutes with arbitrary directed direct limits. Moreover, show that if $M \in \mathcal{M}^f(A)$ then $\text{Ext}^i(M, \cdot)$ commutes with directed direct limits, and $\text{Hom}(M, \cdot)$ commutes with arbitrary direct limits. Do $\text{Ext}^i(M, \cdot)$ commute with arbitrary direct limits?