EXERCISE 6 IN D-MODULES I

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Let $X$ be a smooth algebraic variety and $Z \subset X$ a (Zariski) closed subvariety. Let $i : Z \hookrightarrow X$ denote the embedding.

(1) (P) Show that for any smooth affine variety $X$, $\mathcal{D}^\leq 0(X) = \mathcal{O}(X)$, and $\mathcal{D}^\leq 1(X)/\mathcal{O}(X) \cong \text{Der}\mathcal{O}(X)$.

(2) (*)

(a) If $X = \{\sum_i x_i^2 = 0\}$ then $\mathcal{D}(X)$ is Noetherian but not generated by $\mathcal{D}^\leq 1(X)$.

(b) If $X = \{\sum_i x_i^3 = 0\}$ then $\mathcal{D}(X)$ is not Noetherian.

(3) (P) Let $H \in \mathcal{M}_{\text{coh}}^R(\mathcal{D}_Z)$. Show that $i_0(H)$ is also coherent and $\text{SS}(i_0(H)) = \{(x, \xi) \in T^*_X x \in Z \text{ and } (x, p_{X,x}(\xi)) \in \text{SS}(H)\}$, where $p_{X,x} : T^*_X X \to T^*_Z Z$ denotes the standard projection.

(4) (P) Let $i : \mathbb{A}^1 \to \mathbb{A}^2$ be the standard embedding. Compute $i_0 M$ for

(a) $M = K[x]$

(b) $M = D_1/\partial^5$

(c) $M = D_1/x\partial$

(d) $M = D_1/(x^2\partial + 4x + 1)$

(5) Show that if $X$ is affine then the structure of a left $\mathcal{D}(X)$-module on an $\mathcal{O}(X)$-module $M$ is the same as the action of the Lie algebra $\mathcal{T}(X)$ of algebraic vector fields, that satisfies

(a) $[[\xi_1], \xi_2] = [[\xi_1, \xi_2]]$

(b) $[[\xi], \langle f \rangle] = \langle \xi(f) \rangle$

(c) $\langle f \xi \rangle = \langle f \rangle \cdot \langle \xi \rangle$,

where $\xi, \xi_1, \xi_2 \in \mathcal{T}(X)$, $f \in \mathcal{O}(X)$, and $\langle \cdot \rangle$ denotes the action on $M$.

(6) Let $M \in \mathcal{M}(\mathcal{D}_n)$. Let $\varphi : M \to M[x_n^{-1}]$ be the natural map. Show that ker $\varphi$ and coker $\varphi$ are supported on the hyperplane $\{x_n = 0\}$.

(7) The module of top differential forms $\Omega^\top_X$ with the action $\xi \alpha := -\text{Lie}_\xi \alpha$ (Lie derivative) is a right $\mathcal{D}(X)$-module. Moreover, $M \mapsto M \otimes_{\mathcal{O}_X} \Omega^\top_X$ defines an equivalence of categories $\mathcal{M}(\mathcal{D}(X)) \simeq \mathcal{M}^*(\mathcal{D}(X))$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/DmodI_3.html

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