EXERCISE 3 IN D-MODULES II

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(1) Let $X$ be an irreducible algebraic variety of dimension $n$ and $M$ a coherent $O_X$-module without torsion.

**Definition 1.** We will call a **semi-small extension** of $M$ a coherent $O_X$-module $N$ without torsion that contains $M$, such that the support $\text{supp}(N/M)$ has dimension $\leq n - 1$.

We call this module a **small extension** if $\text{supp}(N/M)$ has dimension $\leq n - 2$.

(i) Show that semi-small extensions $N \supset M$ are naturally realized as coherent $O_X$-submodules of the module $M_K = K \otimes_{O_X} M$, where $K$ denotes the field of rational functions.

In particular they form a partially ordered set.

(ii) Let us set $M^* := \text{Hom}(M, O_X)$. Show that this is a coherent $O_X$-module.

Construct a canonical morphism $i : M \to M^{**}$ and show that $M$ does not have torsion if and only if $i : M \to M^{**}$ is an embedding.

(2) (P) Let $X$ be isomorphic to the affine space $\mathbb{A}^n$.

(i) Let $M$ be a free $O_X$-module. Show that $M$ has no proper small extensions.

**Hint.** Reduce to the case when $X = \mathbb{A}^n$ and $S = \text{supp}(N/M) \subset \mathbb{A}^{n-2}$

(ii) Show that for a semi-small extension $N \supset M$ the natural morphism $N^* \to M^*$ is injective.

Show that for a small extension this morphism is an isomorphism.

(3) (P) Let $X$ be an irreducible algebraic variety. Show that any coherent $O_X$-module $M$ without torsion has a maximal small extension, i.e. there exists a small extension $N \supset M$ that contains all other small extensions.

**Hint.** Reduce to the case when $X \approx \mathbb{A}^n$.

Show that in this case the coherent $O_X$-module $M^{**}$ contains all small extensions of $M$, and itself is a small extension.

URL: [http://www.wisdom.weizmann.ac.il/~dimagur/DmodII.html](http://www.wisdom.weizmann.ac.il/~dimagur/DmodII.html)