13. Lecture 13. Verdier Specialization

13.1. Structure of characteristic variety of a holonomic module. Let $M$ be a holonomic $\mathcal{D}_X$-module, $S = SS(M)$ its singular support.

Corollary of Involutivity Theorem.
$S$ is a Lagrangian subvariety of $T^*X$

Lemma 13.1.1. Let $S$ be a closed irreducible conic Lagrangian subvariety of $T^*X$.

Then there exists a smooth locally closed subvariety $Z \subset X$ such that $S$ is the closure of the conormal bundle $N^*Z \subset T^*X$.

Thus, starting with a holonomic $\mathcal{D}_X$-module $M$, we can construct several irreducible subvarieties $Z_i \subset X$ that describe its singular support.
13.2. **Other approaches to $RS$-modules.**

**Claim.** A holonomic $\mathcal{D}_X$-module $M$ is $RS$ iff there exists a good filtration of $M$ with a property that $\text{gr}(M)$ is strictly supported on $SS(M)$.

This means, that if we denote by $I \subset O_{T,X}$ the ideal of functions that vanish on $SS(M)$ then $I \cdot \text{gr}(M) = 0$.

This lemma is correct but I do not know how to prove it elementary. I would like to describe some way how one can approach this proof.
13.2.1. **Functor of nearby cycles.** There are some functors important defined on holonomic modules that can not be expressed directly in terms of six Grothendieck functors.

Let \( \mathbb{A} \) denote the standard affine line with coordinate \( t \), \( 0 \in \mathbb{A} - \) is a closed subset of \( \mathbb{A} \). We denote by \( i : 0 \to \mathbb{A} \) the closed imbedding, by \( \mathbb{A}^* \) the punctured line \( \mathbb{A} \setminus 0 \) and by \( j \) the open imbedding \( j : \mathbb{A}^* \to \mathbb{A} \).

Consider a holonomic module \( N \) on \( \mathbb{A}^* \) and set \( M = j_*(N) \). This is a module, and it is holonomic.

Let \( M' \) denote the maximal quotient of \( M \) supported at 0. Set \( \Psi(N) := i!(R) \).

Thus we constructed a functor \( \Psi : Hol(\mathbb{A}^*) \to Hol(pt) \) via \( \Psi(N) := i!(R) \).

\[
\begin{align*}
R &= M / \mathcal{L}_0 \{ t^n \} \\
M &\subset L - \text{subspace of} \ L \\
\text{Claim} \Psi \text{is an exact functor.}
\end{align*}
\]
In fact, now we can repeat this construction in more general situation. Let $X$ be an algebraic variety and $t$ a regular function on $X$. We can interpret $t$ geometrically as a morphism $t : X \to \mathbb{A}$.

Set $X_0 = t^{-1}(0)$ and $X^* = t^{-1}(\mathbb{A}^*)$.

Then in the same way we define the functor

$\Psi : Hol(X^*) \to Hol(X_0)$.

13.3. **Verdier construction.** Let $X$ be a smooth variety and $Y \subset X$ a closed smooth subvariety.

Verdier constructs a deformation of $X$ to the variety $N^*Y$ – the total variety of the conormal bundle to $Y$ in $X$.

Namely he considers the algebraic variety $X$ and $t$
namely we constructed an algebraic variety $\mathfrak{A}$ and a morphism $t : Z \to \mathfrak{A}$ such that

(i) $Z_0 = t^{-1}(0)$ is isomorphic to $N^*_y$.

(ii) The variety $Z^* = t^{-1}(\mathfrak{A}^*)$ is canonically isomorphic to $\mathfrak{A}^* \times X$ with equivariant projection $Z^* \to \mathfrak{A}^*$.

Construction.

\[ y = f(x) \quad o(x,y) = 0 \quad o(x,y) = 0 \quad o(x,y) = 0 \]

\[ o(x) = o(x) \quad o(x,y) = 0 \quad o(x,y) = 0 \]

\[ Z = \text{spec } B \]

\[ \mathfrak{A} = \mathfrak{A}_0 \to B \quad Z \to X \]

\[ \text{Sp} : \mathfrak{A}_0 \to \mathfrak{A} \]

\[ \text{Sp} : \mathfrak{A}_0 \to \mathfrak{A} \]

Consider case

\[ x = 0 \quad y = 0 \]

\[ \text{Sp} : (\mathfrak{A}_0) = \text{spec } B, \quad \mathfrak{A}_0 \]

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\[ N^*_y \quad \mathfrak{A}_0 \quad \mathfrak{A} \]

Specialization functor.

\[ \text{Sp} : \mathfrak{A}_0 \to \mathfrak{A}_0 \]

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\[ \text{Sp} : \mathfrak{A}_0 \to \mathfrak{A}_0 \]
\[ \text{Sp}_g(x^2) = 0 \]

Module $D_k$-module $x \leq k$ is a smooth inv. subvariety

(i) $\text{Sp}_g(N) = 0$ if $x$ is a minimal
one of special submanifold for $M$.

(ii) If $x$ is a component then $\dim \text{Sp}_g(N) \leq \text{null}$, and claim. http://rs iff there is an equals