14. Lecture 14. Correction about Verdier specialization functor

14.1. Deformation to the normal cone. Let $X$ be a smooth variety, $Y \subset X$ a closed smooth subvariety. Consider a normal bundle $N_Y$ to the subvariety $Y$ and denote by $N_Y X$ the total space of this bundle.

Informal remark. Consider $Y$ as a subvariety in $X$ and in $N_Y X$ (zero section). Then tubular neighborhoods of $Y$ in these two spaces are very close.

Formally this means that there exists a deformation of the space $N_Y X$ to $X$.

Claim. There exists a smooth algebraic variety $Z$ with the action of $G_m$ and a morphism $p : Z \to \mathbb{A} = \mathbb{A}^1$ such that

(i) $p$ is $G_m$ equivariant.

(ii) $Z_0 = p^{-1}(0)$ is isomorphic to $N_Y X$ compatible with the action of $G_m$.

(iii) The complement $Z^* = p^{-1}(\mathbb{A}^*)$ is isomorphic to $G_m \times X$ compatible with the action of $G_m$. 
Construction of the deformation $Z$.

Consider the variety $W = \mathbb{A} \times X$ and a subvariety $Y = Y \times 0 \subset W$. The variety $W$ has natural $G_m$ action.

We define $Z'$ to be a blow-up of $W$ at $Y$, $p : Z' \to Y$ the natural projection.

We get $Z$ by removing from $Z'$ closed subset that is blow-up of $0 \times X$ at $Y$.

14.1.1. Nearby cycles. Let $p : Z \to \mathbb{A}$ be a projection. denote by $t$ the corresponding function on $Z$.

We define the functor $\Psi : \text{Hol}(Z^*) \to \text{Hol}(Z_0)$

Starting with a holonomic module $M$ consider the module $M' = M \cdot t^*$ over the ring $k[[s]]$.

Then we set $\psi(M) = \text{Conc}(j_!(M')) \to j_*(M')$.

This functor is also defined on the category $D_h(D_X)$. It is an exact functor.
**Definition.** Let $X$ be a smooth variety and $Y \subset X$ a smooth subvariety. We define the **Specialization** functor

$$Sp : D_h(X) \to D_h(N_Y X \mathfrak{a})$$

as $Sp(M) = \Psi \circ q^1(M)$, where $q : Z^* \to X$ is the projection.

Let $G_W$ act on variety $W$ by $D_W$-modules. Suppose $G_W$ acts on $W$ as a $D_W$-module and acts with order as a $D_W$.

$D_W; M \to M$ is $G_W$-equivariant if $D_W$-module is called weakly $G_W$-equivariant.

Let $G_W$ be the standard generator.

The action of $G_W$ is

1. $(f)\text{-derivatives of the}$
   - action of $G_W$
2. $\xi \mapsto \text{action of variety}$

$\text{def} = \tau(f) - q(f) : V^* \to M$ is a map of $D$-modules.

**Cospecialization**

$$\text{cospecialization}$$

$$\text{cospecialization} : \text{D}_h(D(W)) \to \text{D}_h(N^\alpha_Y X).$$

**Fourier:**

$$D(N^\alpha_Y X) \to D(R^\alpha_Y X).$$

**Let $V \to X$ be a vector bundle.**

**Fourier:**

$$D(TV) \to D(TV^*)$$

$\cdots$ $\cdots$ $\cdots$

**veeb $\mathfrak{a}$, $X \to X^*$

$T^* \cong \text{cotangent bundle}$

$D(V) = k[x_1, \ldots, x_n]$

$\text{D}(\mathfrak{a}) = k[x_1, y_1]$

$x = 0$. 
Consider the line $\ell$, a vector $\mathbf{v}$, and a function $f : \ell \to \mathbb{R}$.

Let $L = \{ v | \theta \in \ell \}$.

Set $V = \mathbb{R}^3$.

For $L$, $L^* = \mathbb{R}^3$.

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Claim: On weekly semi-modules, modules form non-abelian.

Consider $D(\mathbf{x}) \to D^*_{\mathbf{v}}(\mathbf{u}^*, \mathbf{x})$

Suppose $\mathbf{v} = \mathbf{u}^* (\mathbf{v}, \mathbf{x})$

Then for any $\mathbf{y} \in \mathbf{v}$

$\mathbf{y} \notin (\mathbf{v}) \subseteq D^*_{\mathbf{v}}(\mathbf{u}^*, \mathbf{x})$.

Or $M(\mathbf{u}^*, \mathbf{x}) \ni (\mathbf{v}^*, \mathbf{x})$. 

(i) $\mathbf{v} \notin (\mathbf{v}^*)$ for any $\mathbf{y}$.

These values are equal.
Colom's theorem

Definition. An $s$-algebra is an $s$-algebra $A$ with $1$ and $c$ in $s$.