3. Lecture 3. RS in high dimension

We have discussed the notion of RS in dimensions ≤ 1. How to define the notion of RS in higher dimensions?

Recall, that in case of holonomic modules and complexes we could give a definition using restrictions to points. So we can define RS using restrictions to curves. Later we will discuss other approaches.

Let X be an algebraic variety. A test curve on X is a morphism \( \nu : C \to X \), where C is a smooth curve.

**Definition.** A \( \mathcal{D} \)-complex \( F \) on \( X \) is called **RS** if it is holonomic and for any test curve \((C, \nu)\) the restriction \( \nu'(F) \) is RS on the curve \( C \).

We denote by \( \mathcal{D}_{RS}(\mathcal{D}_X) \) the category of RS-complexes (a full subcategory of \( \mathcal{D}(\mathcal{D}_X) \)).

A \( \mathcal{D}_X \)-module is called RS if it is RS as a \( \mathcal{D}_X \)-complex. These modules form a full subcategory \( RS(\mathcal{D}_X) \) of \( M(\mathcal{D}_X) \).

Discussion – Pro and contra of this definition.
Our goal is to show the following

**Theorem 3.1.**

1. Subcategory $D_{RS}$ is a triangulated subcategory closed under extensions.
2. Categories of $RS$ complexes are preserved by all functors
   \[ \pi^1, \pi^*_1, \pi^*, \Pi, \mathcal{D}. \]
3. A $D_X$-complex $F$ is $RS$ iff all its cohomology modules are $RS$.
4. The subcategory $RS(D_X) \subset M(D_X)$ is an abelian subcategory closed with respect to subquotients and extensions.

---

3.2. $RS$ for smooth modules. We know that holonomic complexes can be generated by images of smooth $D$-modules. So it is natural to study the notion $RS$ first for smooth $D$-modules.

Let $X$ be a smooth variety, $E$ a smooth $D$-module on $X$. We can think about $E$ as a vector bundle with a flat connection $\nabla$.

For any test curve $\nu : C \to X$ we see that $D_C$-complex $\nu^!(E)$ up to cohomological shift coincides with the vector bundle $\nu^*(E)$ with induced connection. Hence $E$ is $RS$ iff it satisfies the following condition

(1) $\nu^!(E)$ for any test curve $(\nu, C)$ the bundle $\nu^!(E)$ on $C$.
For any test curve $(\nu, C)$ the bundle $\nu^*(E)$ on $C$ is $RS$.

Let us consider slightly more general situation.

3.2.1. Regular singularity along a closed subset $S$. Let $X$ be a smooth algebraic variety of dimension $n$, $S \subset X$ a closed subset (usually it will be a divisor).

Set $U = X \setminus S$ and denote by $j : U \to X$ the open imbedding.

Let $E$ be a smooth $\mathcal{D}_U$-module. We would like to define a notion that $E$ is $RS$ along the subset $S$.

In this situation we consider pointed test curves. Namely, this is a pointed smooth curve $(C, s)$ equipped with a morphism $\nu : C \to X$ such that $\nu(s) \in S$ and $\nu(C \setminus s) \subset U$.

We say that $E$ is $RS$ along $S$ if it satisfies the following condition:

(RS) For any pointed test curve $(\nu, C, s)$ the bundle $\nu^*(E)$ on $C \setminus s$ is $RS$ at the point $s$.

In the study of smooth $RS$-modules important role is played by the following informal principle:

Principle. If the condition $RS$ holds for many pointed test curves then it holds for all pointed test curves.
3.2.2. *RS along smooth divisor S*. Let us consider the important case when $X$ is smooth and $S \subset X$ is a smooth divisor. We denote by $\mathcal{D}_{X,S}$ the sheaf of subalgebras in $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and by vector fields tangent to $S$.

Locally we can choose coordinate system $x_1, \ldots, x_n$ on $X$ such that $S$ is defined by equation $t = 0$, where $t = x_n$. Then the algebra $\mathcal{D}_{X,S}$ is generated by $\mathcal{O}_X$ and vector fields $\partial_i$ for $i = 1, \ldots, n-1$ and $d = t \partial_n$.

Let $E$ be a smooth $\mathcal{D}_U$-module, where $U = X \setminus S$. We set $F := j_*(E)$.

**Definition. 1.** We call an *S-lattice in F* a coherent $\mathcal{O}_X$-submodule $E'$ such that the restriction of $E'$ to $U$ coincides with $E$.

2. We say that the $S$-lattice $E'$ is **admissible** if is $\mathcal{D}_{X,S}$-invariant.

3. We say that the smooth $\mathcal{D}$-module $E$ is *algebraically RS along S* if the sheaf $F$ has an admissible $S$-lattice.

It is easy to prove the following

**Lemma 3.2.3.** (i) Any two $S$-lattices $E', E''$ are (locally) $t$-equivalent, i.e. there exists a number $N$ such that $E'' \subset t^{-N}E'$ and $E' \subset t^{-N}E''$.

(ii) If $F$ has an admissible $S$-lattice, then any $\mathcal{O}_X$-coherent subsheaf $E' \subset F$ is contained in an admissible $S$-lattice.

\[
\mathcal{X} = \mathcal{A}^1, \quad \mathcal{S} = \{0\}. \quad \mathcal{U} = \mathcal{A}^1.0
\]

$E$ corresponds to function $f = e^{1/t}$ not regular at $t=0$. $f$
We will prove the following key criterion of $RS$.

**Proposition 3.2.4.** $E$ is algebraically $RS$ along $S$ iff it is $RS$ along $S$, i.e. its restriction to any test curve is $RS$.

**Corollary 3.2.5.** Let $S$ be a smooth divisor.

Suppose there exists an open dense subset $S' \subset S$ and its open neighborhood $W$ in $X$ such that the restriction of the smooth $\mathcal{D}_X$ module $E$ to $W$ is $RS$ along $S'$. Then $E$ is $RS$ along $S$.

**Proof**
\( H = \mathfrak{H}(E) \)

\[ \mathcal{H} \subset \mathcal{E} \]

\( i : V \to X \)

\( F = i^*(E) \)

**Claim.** (i) \( H \) is \( D_{\text{sh}} \)-linear

(ii) \( \mathcal{H} \) is \( \mathcal{D}_{\text{sh}} \)-coherent

(iii) \( \mathcal{H} \) is an admissible

\[ \mathcal{H} \Rightarrow E \in \mathcal{A}_{\text{sh}} \]

**Proof of i.**

(i) \( \mathcal{H} \) does not have torsion.

(ii) \( \mathcal{H} \) is coherent on \( V \) and on \( W \)

\[ \mathcal{X} \cap (V \cup W) \text{ has dim} > 1 \]

**Lemma:** Let \( V \) be a subspace.

\( D_{\text{sh}} \)-module \( V \)-twisted.

Suppose \( \mathcal{H} \) is without torsion

and \( \mathcal{H} \) is coherent

outside of a closed subset \( T \) of \( \mathcal{H} \)?

Then \( \mathcal{H} \) is coherent

\[ V = X \setminus T \quad \mathcal{V} : V \to X \]

\[ \mathcal{H} \quad (\mathcal{H} \subset \mathcal{V}^*(\mathcal{A}_V)) \]

**Sublemma:** \( V \subset X \)-open.

\( R \) is \( D_{\text{sh}} \)-coherent \( \mathcal{A}_V \)-module without torsion

\[ V = X \setminus T \quad \text{closed} \]

Then \( \mathcal{V}^*(R) \) is coherent

(i) \( \text{case } R = \mathcal{O}_V \)

\[ \mathcal{O}_X \subset \mathcal{O}_V \]

\( \text{Functions } f : R \to \mathcal{O}_V \)

\( \text{separate elements } \mathcal{R} \)

\( \text{Finite number of } f \)

\( R \subset \mathcal{O}_V \).
3.2.6. **Divisor with normal crossings.**

\[ S \subset X \]

A divisor with normal crossings is defined in a coordinate system \( \mathbb{K}^n \) such that \( S \) is given by the equation \( x_1 x_2 \cdots x_n = 0 \).

**Claim:** Convexity holds in this case.

**Example:** \( S \subset \mathbb{K}^1 \)

\[ S \]

**Proof:**

\( D_{x_1} \) is a divisor of codimension \( n \) generated by \( \partial x_1 \) and vector fields tangent to \( S \).

Locally, it is generated by \( \partial x_1, \partial x_2, \ldots, \partial x_n, \)

\[ \partial z_i = \partial x_i \partial z_i \quad i = 1, 2, \ldots, n \]

**Property:** \( E \) is contractible.
\( RS \) along \( s \) is equivalent to \( RS \) along \( s \).

**Proof.**

We have an admissible lattice \( E' \subset T \).

Consider curve \((C, \gamma)\) with parameter \( t \in T \).

\[ x_i = \gamma_i(t) \text{ for } i = 1, \ldots, n. \]

Define vector field \( \mathcal{E}_C \) on \( C \) such that it is extended to a vector field \( E' \mid_C \) on \( C \subset E' \).

This implies that if \( E' \mid_C \) is an admissible lattice in \( E' \),

**Deligne’s criterion**

\( \mathcal{E} \subseteq E' \) where \( E' \) is a vector field.