LECTURE 5 IN D-MODULES II - PROPERTIES OF ALGEBRAIC-RS MODULES

Let $U$ be a smooth quasi-projective algebraic variety, and $X$ be its good compactification. This means that $X$ is projective and $S := X \setminus U$ is a divisor with strict normal crossings.

Definition 1. Let $\mathcal{D}_{X,S} \subset \mathcal{D}_X$ denote the sheaf of subalgebras generated by $\mathcal{O}_X$ and by vector fields tangent to $S$.

A coherent $\mathcal{D}_X$-module $\mathcal{F}$ is called algebraic RS (with respect to $U$) if its restriction $\mathcal{F}|_U$ is smooth, and $\mathcal{F}$ is a union of $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-submodules.

In this lecture we will prove several claims on algebraic RS-modules that were left from before.

Exercise 2. The category of algebraic RS-modules is closed w.r. to subquotients.

Proposition 3. Let $\mathcal{E}$ be a smooth RS $\mathcal{D}_U$-module. Then $j_* \mathcal{E}$ is algebraic RS (with respect to $U$).

In the proof we will use the following statements.

Exercise 4. $\mathcal{F}$ is algebraic RS if and only if every coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ generates an $\mathcal{O}_X$-coherent $\mathcal{D}_{X,S}$-submodule.

Lemma 5. The proposition holds if $S$ is smooth, and without the assumption that $X$ is projective.

This lemma will be proven in later weeks.

Exercise 6. Let $Y$ be an algebraic variety, and $V \subset Y$ and open subset such that $\dim Y \setminus V \leq \dim Y - 2$. Let $\mathcal{H}$ be an $\mathcal{O}_Y$-module without torsion.

(i) If $\mathcal{H}|_V$ is coherent then $\mathcal{H}$ is coherent.

(ii) Let $\mathcal{H}_1 \subset \ldots \mathcal{H}_i \subset \mathcal{H}_{i+1} \subset \ldots \subset \mathcal{H}$ be an increasing sequence of coherent submodules. If the sequence of restrictions $\mathcal{H}_i|_V$ stabilizes then so does the original sequence $\mathcal{H}_i$.

Proof of Proposition 3. Let $Z$ denote the singular locus of $S$, and let $V := X \setminus Z$. Let $\mathcal{F}' \subset \mathcal{F} := j_* \mathcal{E}$ be an $\mathcal{O}_X$-coherent submodule, and let $\mathcal{H} \subset \mathcal{F}$ be the $\mathcal{D}_{X,S}$-submodule generated by $\mathcal{F}'$. Then $\mathcal{H}|_V$ is $\mathcal{O}_V$-coherent by the smooth case - Lemma 5. Thus by Exercise 6 $\mathcal{H}$ is $\mathcal{O}_X$-coherent. □

Corollary 7. Let $\nu : j_! \mathcal{E} \to j_* \mathcal{E}$ be the natural morphism. Then the modules $L(S, \mathcal{E}) := \text{Im}(\nu)$, and $\text{Coker} \nu$ are algebraic RS.

Proposition 8. Let $Z$ be a smooth affine curve, $Y = Z \times \mathbb{A}^1$, $V \subset Y$ open dense, $p : V \to Z$ the natural projection. Let $\mathcal{E}$ be a smooth RS $\mathcal{D}_V$-module. Then $p_* \mathcal{E}$ is RS on $Z$. 

To prove this we may restrict to an open subset \( Z' \) of \( Z \), and its preimage \( U \) in \( V \). We can find a good compactification \( X \) of \( U \), mapping onto the completion \( C \) of \( Z' \), such that the inverse image of \( T = C \setminus Z' \) is contained in \( S = X \setminus U \), and that \( T \) contains the image of the singular locus of \( S \). By adding some points of \( C \) to \( T \), and their preimages to \( S \), if needed, we may further arrange that any component of \( S \) which maps onto \( C \) is an unramified covering outside its intersection with the singular set of \( S \). Now let \( j : U \to X \) denote the inclusion and \( \pi : X \to C \) denote the projection. Let \( \mathcal{F} := j_* \mathcal{E} \). We know that \( \mathcal{F} \) is algebraic RS. Thus it is enough to prove the following proposition.

**Proposition 9.** Let \( \mathcal{F} \) be a \( \mathcal{D}_X \)-module that is algebraic RS (with respect to \( U \)). Then \( \pi_* \mathcal{F} \) is algebraic RS on \( C \) (with respect to \( Z \)).

To prove this proposition we need to construct lattices on the cohomologies of \( \pi_* \mathcal{F} \). We do so by constructing pushforward for \( \mathcal{D}_{X,S} \)-modules. We will show that this construction will preserve \( \mathcal{O} \)-coherence since \( \pi \) is projective.

Let \( \theta_{X,S} \) denote the vector fields on \( X \) that at points of \( S \) are tangent to \( S \). Similarly, let \( \theta_{C,T} \) denote vector fields on \( C \) that vanish at the points of \( T \). At each point \( x \in X \), the differential \( \pi* \) maps \( T(X)_x \) into \( T(C)_x \). Thus we have a canonical morphism of \( \mathcal{O}_X \)-modules \( \nu : \theta_{X,S} \to \pi* \theta_{C,T} \).

**Lemma 10.** \( \nu \) is onto and its kernel is the module \( \theta_{X/C} \) consisting of germs of vector fields tangent to the fibers of \( \pi \).

**Proof.** The statement on the kernel is easy. Let us prove that \( \nu \) is onto. This statement is local on \( C \). Over \( Z \) it is clear, since on the preimage of \( Z \) \( \pi \) is a submersion. Now let \( x \in \pi^{-1}(T) \). We may choose a local coordinate \( t \) on \( C \) around \( \pi(x) \), and local coordinates \( u, v \) on \( X \) around \( x \) such that \( t \circ \pi = a(u,v)u^m v^n \), where \( a(u,v) \) is regular and invertible around \( x \), and \( m > 0, n \geq 0 \). Around \( x \), the \( \mathcal{O}_C \)-module \( \theta_{C,T} \) is spanned by \( t \partial_t \). Thus it suffices to lift \( t \partial_t \), i.e. to find \( \xi \in \theta_{X,S} \) such that \( \xi(t \circ \pi) = t \circ \pi \). Note that around \( x \), the \( \mathcal{O}_X \)-module \( \theta'_{X,S} \) contains \( u \partial_u \). Take
\[
\xi = (m + a^{-1}(u \partial_u a))^{-1} u \partial_u
\]
Since
\[
u \xi(t \circ \pi) = \xi(a(u,v)u^m v^n) = a(u,v)u^m v^n = t \circ \pi, \text{ so } \xi \text{ is a lift of } t \partial_t. \]

**Proof of Proposition 9.** For every \( \xi \in \theta_{X,S} \), there exist \( u_i \in \mathcal{O}_X \) and \( \xi_i \in \theta_{C,T} \) by
\[
\nu(\xi) = \sum_i u_i \otimes \pi^{-1} \xi_i
\]
For every left \( \mathcal{D}_{C,T} \)-module \( M \), let \( \pi^0(M) \) denote its pullback to \( X \) as an \( \mathcal{O} \)-module, with a \( \mathcal{D}_{X,S} \)-module structure given by
\[
\xi(f \otimes m) = \xi f \otimes m + \sum_i f \cdot u_i \otimes \pi^{-1}(\xi_i m)
\]
Let \( \mathcal{D}_{X,S-C,T} := \pi^0(\mathcal{D}_{C,T}) \), endowed with the natural left \( \mathcal{D}_{X,S} \)-module structure. For every right \( \mathcal{D}_{X,S} \)-module \( N \) define \( \pi_+ M := R\pi_+(N \otimes^L \mathcal{D}_{X,S-C,T}) \), where \( \pi_+ \) refers
to the morphism of ringed spaces \((X, \pi^{-1}D_{C,T}) \to (C, D_{C,T})\), and \(R\) denotes the right derived functor. It is enough to show that \(\pi_*\) maps \(\mathcal{O}_X\)-coherent \(D_{X,S}\)-modules to complexes with \(\mathcal{O}_C\)-coherent cohomologies. To do this note that the module \(D_{X,S} \to D_{X,S} \to D_{X,S} \to C, T \to 0\) admits a locally free resolution

\[ 0 \to \theta_{X/C} \otimes_C D_{X,S} \to D_{X,S} \to D_{X,S} \to C, T \to 0 \]

Thus

\[ N \otimes_{D_{X,S}} D_{X,S} \to C, T = \{ N \otimes_C \theta_{X/C} \to N \}, \text{ with differential } u \otimes \xi \mapsto u\xi \]

Since it consists of \(\mathcal{O}_X\)-coherent modules and \(\pi\) is projective, the direct image of this complex has \(\mathcal{O}_C\)-coherent cohomologies. □

**Proposition 11.** Let \(U\) be a smooth (quasi-projective) algebraic variety, and \(X\) be a good completion of \(U\). Let \(\mathcal{F}\) be \(D_X\)-module. If \(\mathcal{F}\) is algebraic RS with respect to \(U\) then \(\mathcal{F}\) is RS.

**Proof.** We have to show that for every smooth projective curve \(C\), and every morphism \(\nu : C \to X\), the inverse image \(\nu^*\mathcal{F}\) is RS. For this we may assume that \(V := \nu^{-1}(U)\) is open dense in \(C\), and we have to construct a lattice around every point \(c \in C \setminus V\).

We construct the lattice by pulling back an \(D_X\)-coherent \(D_{X,S}\) submodule \(\mathcal{H} \subset \mathcal{F}\) that satisfies \(\mathcal{H}|_U = \mathcal{F}\). To show that it is a lattice, let \(t\) be a local coordinate at \(c\), and \(d = t\partial_t\). We have to show that \(d \in \nu^*D_{X,S}\).

If \(S\) is smooth at \(\nu(c)\), let \(x\) be a local coordinate at \(\nu(c)\) transversal to \(S\). Then \(x = a(t)t^n\), where \(n > 0\) and \(a(t)\) is invertible near 0, and \(d = \xi := (n + a^{-1}(t)(t\partial_t a(t)))x \partial_x\). Indeed, \(\xi x = (t\partial_t a(t) + na(t))t^n\), and thus \(\xi = d = t\partial_t\).

If \(S\) is not smooth at \(\nu(c)\), it still has normal crossings. Suppose it has 2 components near \(\nu(c)\), with local coordinates \(x\) and \(y\). Then \(x = a(t)t^n\), and \(y = b(t)t^m\). Take \(\xi := (n + a^{-1}(t)(t\partial_t a(t)))x \partial_x + (m + b^{-1}(t)(t\partial_t b(t)))y \partial_y\)

□