7. LECTURE 7. RIEMANN - HILBERT CORRESPONDENCE.

In this lecture we will consider algebraic geometry over the field \( \mathbb{C} \) of complex numbers.

Given a smooth algebraic variety \( X \) over \( \mathbb{C} \) we can consider the set of points \( X(\mathbb{C}) \) as a topological space in usual – analytic – topology. We denote this space by \( X_{\text{top}} \).

This space in fact is an oriented manifold. Our general goal is to understand how to connect topological invariants of this space with algebraic invariants of the variety \( X \). Let me briefly discuss what we can expect.

Let \( Sh(X_{\text{top}}) \) denote the category of sheaves of complex vector spaces on \( X_{\text{top}} \). We will see that many invariants defined in terms of this category have an algebraic counterpart.

To study these invariants we will use an intermediate object – the complex analytic variety \( X_{an} \). This is the topological space \( X_{\text{top}} \) equipped with the sheaf \( O_{an} \) of analytic functions.
7.1. Local systems.

Definition. A local system (of complex vector spaces) on the topological space $X_{\text{top}}$ is a sheaf of complex vector spaces that is locally isomorphic to the constant sheaf $\mathbb{C}^n$ for some $n$.

We denote by $\text{LocSys}(X_{\text{top}})$ the full subcategory of local systems of the category $\text{Sh}(X_{\text{top}})$.

Remark. Local systems and representations of the fundamental group.

\[
\text{\textit{X}-connected manifold, box } \mathcal{T}_1, (X, \mathbb{C}^n) \\
\text{L-local system on X} \\
\text{L(\mathfrak{g})-fiber at } x \text{ then we get an action of } \mathfrak{gl}(\mathbb{C}^n) \\
on \mathcal{T}_1, \text{ on } L(x, \mathbb{C}^n). \\
\text{Loc } L(x, \mathbb{C}^n) \cong \text{Rep } \mathfrak{gl}(\mathbb{C}^n(x, \mathbb{C}^n))
\]

The category $\text{LocSys}(X_{\text{top}})$ can be described in algebraic terms. First state the following easy result.

Proposition 7.1.1. Let $X_{\text{an}} = (X_{\text{top}}, O_{\text{an}})$ be a smooth complex analytic variety over $\mathbb{C}$. The category $\text{LocSys}(X_{\text{top}})$
of local systems on the space $\mathcal{X}_{\text{top}}$ is canonically equivalent to the category $\text{Smooth}(X_{\text{an}})$ of analytic vector bundles $E$ on $X_{\text{an}}$ equipped with a holomorphic flat connection $\nabla$.

The mutually inverse functors here are given by

$L \mapsto O_{\text{an}} \otimes_{\mathbb{C}} L$

$(E, \nabla) \mapsto$ sheaf of flat sections of $E$.

Let $X$ be a projective smooth algebraic variety over $\mathbb{C}$. We will see later that in this case the category of algebraic vector bundles on $X$ is canonically equivalent to the category of analytic vector bundles on the analytic variety $X_{\text{an}}$. From this it is easy to deduce the following

**Corollary 7.1.2.** Let $X$ be a smooth projective algebraic variety over $\mathbb{C}$. Then the category $\text{LocSys}(X_{\text{top}})$ is canonically equivalent to the category $\text{Smooth}(\mathcal{D}_X)$ of smooth $\mathcal{D}_X$-modules.

In fact, this is true for any complete smooth variety. We will prove this later and also we will show how to compute the cohomology of local systems in algebraic terms.

However, in the case when $X$ is not complete the situation is more complicated.
Example 7.1.3. Let $X = A^1$. consider two smooth $\mathcal{D}_X$-modules of dimension 1 over $\mathcal{O}_X$

$$M = \mathcal{D}_X / \mathcal{D}_X \cdot \partial_t$$
and
$$N = \mathcal{D}_X / \mathcal{D}_X \cdot (\partial_t - 1).$$

These modules are generated by functions $f = 1$ and $h = \exp(t)$.

Analytically these modules are isomorphic, and they both correspond to the trivial local system on $X_{\text{top}}$. But as algebraic $\mathcal{D}_X$-modules they are not isomorphic.

7.2. How to state the Riemann–Hilbert correspondence. The classical Riemann–Hilbert correspondence is the following

Theorem 7.3. Let $X$ be a smooth algebraic curve. Then the category $\text{LocSys}(X_{\text{top}})$ is canonically equivalent to the category of smooth $\mathcal{D}_X$-modules.

This theorem has natural generalization for an arbitrary smooth algebraic variety $X$ – in fact, it is formulated in exactly the same way.

However, in 1976 Kashiwara conjectured that some much more general claim has to be true. Kashiwara and Kawai have proven some form of this conjecture, but their proof was very heavy.

The reason was that they worked in the framework of $\mathcal{D}$-modules on complex analytic varieties, and this theory is much less structured than the theory of algebraic $\mathcal{D}$-modules. For example, for such modules there is no natural general notion of direct image.
We will work in algebraic situation. Let us first introduce the objects that appear in RH correspondence.

7.4. Constructible sheaves. Let $X$ be an algebraic variety. Let us remind that a stratification of $X$ is a finite collection $\mathcal{S} = (S_i)$ of smooth algebraic subvarieties $S_i \subset X$ - they are called strata of $\mathcal{S}$ - such that they are disjoint and cover $X$.

Usually one adds a condition that the closure of every stratum is a union of strata and some other conditions. However for our purposes this is not important, since any stratification can be refined to a stratification satisfying these conditions.

Let $X_{top}$ be the topological space corresponding to $XR$ and $Sh(X)$ the category of sheaves of complex vector spaces on $X_{top}$.

This category is too large. We will introduce some natural subcategory of it – the category of constructible sheaves, that better describes the geometry.

**Definition.** A sheaf $F \in Sh(X)$ is called constructible if there exists a stratification $\mathcal{S} = (S_i)$ of the algebraic variety $X$ such that the restriction of $F$ to any strata $S_i$ is a local system on $(S_i)_{top}$.

We denote by $Con(X) \subset Sh(X)$ the full subcategory of constructible sheaves.

Note that this definition refers to the algebraic variety $X$, and not only to the topological space $X_{top}$. 
7.5. Derived category of constructible complexes.
As usual we have to pass from sheaves to derived category.

We denote by $D(X_{top})$ the bounded derived category of the category $Sh(X)$. A complex $F^\cdot$ in this category is called constructible if all its cohomology sheaves are constructible.

We denote by $D_{con}(X)$ the full subcategory of $D(X_{top})$ of constructible complexes.

Thus to every algebraic variety $X$ over $\mathbb{C}$ we have assigned a triangulated category $D_{con}(X)$. This is one of the main objects of study in algebraic geometry.

We will see that this assignment has six Grothendieck functors (in fact, they originally were defined in this situation). Namely, for every morphism $\pi : X \to Y$ of algebraic varieties we will describe exact functors

$$\pi_\ast, \pi_! : D_{con}(X) \to D_{con}(Y)$$

$$\pi^! , \pi^* : D_{con}(Y) \to D_{con}(X)$$

Contravariant functor of Verdier duality $\mathbb{D} : D_{con}(X) \to D_{con}(X)$

Exterior tensor product functor $\boxtimes : D_{con}(X) \times D_{con}(Y) \to D_{con}(X \times Y)$

We will define a functor $\Omega : D(X) \to D(X_{top})$ and prove the following statements

1. Functor $\Omega$ defines an equivalence of triangulated categories $\Omega : D_{RS}(\mathcal{D}_X) \to D_{con}(X)$

2. Functor $\Omega$ is compatible with six Grothendieck functors
3. Let $E$ be a smooth $RS$ $\mathcal{D}_X$-module on a smooth variety $X$ of dimension $n$. Let $L(E)$ be the corresponding local system on the topological space $X_{\text{top}}$.

Then $\Omega(E) = L(E)[n]$ (cohomological shift).

This is the Kashiwara's version of the RH correspondence.

Explanation why we should expect a cohomological shift.

\[\text{Functor in topological situation.}\]
\[\begin{align*}
\pi : X & \to Y \text{ continuous map} \\
\mathcal{F} & \mapsto \mathcal{F} \pi^* X \\
\mathcal{F} & \mapsto \mathcal{F} \pi^* Y
\end{align*}\]

Left exact functor.
\[\mathcal{T} : \mathcal{D}(X) \to \mathcal{D}(Y)\]
\[\mathcal{D}(X) \simeq \mathcal{D}^b(\text{sh}(X)).\]
Functor $T_1$ has left adjoint $\pi' : Sh(Y) \to Sh(X)$

$\pi'$ is exact.

$T_1^* \circ R_1^* = \mathcal{D}(Y \to \mathcal{D}(X))$.

$T_1^* : Sh(Y) \to Sh(Y)$

$\pi'$ on $X$

$\pi'_U(\mathcal{F}) \subseteq \mathcal{F}(U)$. 

\[
\pi''(\mathcal{F}) = \{ \xi \in \mathcal{F}(\pi''_U) \}
\]

Supp $\pi''_U$ is a proper map.

$\pi''_U : Sh(U \setminus \partial U) \to Sh(U)$

$\pi''_U$ is left exact

$T_1^* \circ R_1^* = \pi''_U$

canonical morphism

$T_1^* \to T_1$

$1$ is isomorphism if $\pi''_U$ proper

$\pi''_U : \mathcal{D}(Y) \to \mathcal{D}(X)$ is a right adjoint

extension for $\pi''_U$
\[ T \cdot T = T \cdot T \]

\[ T : X \to Y \text{ complex embedding} \]
\[ T : X \to Y \text{ complex embedding} \]
\[ T \cdot M = \{ z \in H \mid \Re z \leq 0 \} \]

\[ \mathfrak{sl}(Y) \to \text{SL}(X) \text{ left centr.} \]
\[ \mathfrak{sl}(Y) \to \text{SL}(X) \text{ left centr.} \]
\[ X = Y \times R^n \to Y \]
\[ \mathfrak{sl}(Y) \text{ left centr.} \]

\[ \mathfrak{d}(R^n) \text{ is a constant shear in } \mathbb{R}^2 \]
\[ \text{di} \text{normalize for} \]
\[ \mathfrak{p} : \mathbb{R}^n \to \mathfrak{p}^+ \]
\[ \mathfrak{p} : (T \mathfrak{p}^+ \mathfrak{g}) \cong C \]

\[ D : D(X) \to D(LX) \]
\[ \text{Dual}(X) = \mathfrak{p}^+ (LX) \subset D(X) \]
\[ \mathfrak{p} : X \to \mathfrak{p}^+ \]
\[ \mathfrak{d}(F) = \mathbb{R} \times \mathbb{R} \times (F, \text{Dual}(A)) \]

\[ \text{Dual}(X) = \mathfrak{p}^+ (\text{Dual}(X)) \]

\[ \text{Theorem:} \]
\[ \text{If } T : X \to Y \text{ and } \]
\[ \text{then the } \]
\[ \text{then} \]

\[ T \cdot T ; T^*; T^* \text{ preserve constant b} \]
\[ D, \quad \text{open neighborhood of } x \Rightarrow y \\
\text{with } y \in \text{neighborhood of } x \]

\[ N: D(Px) \rightarrow D(Kx). \]