1. Lecture 1: The space of generalized functions on $\mathbb{R}^n$

1.1. Motivation. One of the most basic and important examples of a generalized function is the Dirac delta function. The Dirac delta function on $\mathbb{R}$ at point $t$ is usually denoted by $\delta_t$, and while it is not a function, it can be intuitively described by
\[\delta_t(x) := \begin{cases} \infty & x = t \\ 0 & x \neq t \end{cases},\]
and by satisfying the equality $\int_{-\infty}^{\infty} \delta_t(x) \, dx = 1$. Notice that it also satisfies:
\[\int_{-\infty}^{\infty} \delta_t(x) f(x) \, dx = f(t) \int_{-\infty}^{\infty} \delta_t(x) \, dx = f(t).\]

The following are possible motivations for generalized functions:
- Every real function $f : \mathbb{R} \to \mathbb{R}$ can be constructed as an (ill-defined) sum of $\aleph$ indicator functions $f := \sum_{t \in \mathbb{R}} f(t) \delta_t$.
- In general, solutions to differential equations, and even just derivatives of functions are not functions, but rather generalized function. Using the language of generalized functions allows one to rigorize such notions.
- Generalized function are extremely useful in physics. For example, the density of a point mass can be described by the Dirac delta function.

1.2. Basic definitions. In order to define what is a generalize function we first need to introduce some standard notation.

**Definition 1.2.1** (Smooth functions of compact support).

(i) Denote by $C^\infty(\mathbb{R})$ the space of smooth functions $f : \mathbb{R} \to \mathbb{R}$, i.e. functions that can be differentiated infinitely many times.

(ii) Define the support of a function $f : \mathbb{R} \to \mathbb{R}$ by
\[\text{supp}(f) := \{x \in \mathbb{R} : f(x) \neq 0\},\]
the closure of the set in which it does not vanish.

(iii) Denote by $C^\infty_c(\mathbb{R}) \subset C^\infty(\mathbb{R})$ the space of compactly supported smooth functions.

**Definition 1.2.2** (Convergence in $C^\infty_c(\mathbb{R})$). Given $f \in C^\infty_c(\mathbb{R})$ and a sequence $\{f_n\}_{n=1}^\infty$ of smooth functions with compact support we say that $\{f_n\}_{n=1}^\infty$ converges to $f$ in $C^\infty_c(\mathbb{R})$ if:

Date: November 13, 2017.
(1) There exists a compact set $K \subset \mathbb{R}$ for which $\bigcup_{n \in \mathbb{N}} \text{supp}(f_n) \subseteq K$.

(2) For every order $k \in \mathbb{N}_0$, the derivatives $\left(f^{(k)}_n\right)_{n=1}^\infty$ converge uniformly to the derivative $f^{(k)}$.

We can now define the notion of distributions, we follow [Kan04, Section 2.3]

**Definition 1.2.3 (Distributions).** A linear functional $\xi : C^\infty_c(\mathbb{R}) \to \mathbb{R}$ is continuous if for every convergent sequence $\{f_m\}_{m=1}^\infty$ of functions $f_m \in C^\infty_c(\mathbb{R})$ we have

$$
\lim_{m \to \infty} \langle \xi, f_m \rangle = \langle \xi, \lim_{m \to \infty} f_m \rangle.
$$

We will usually use the notation $\langle \xi, f \rangle$ instead of $\xi(f)$. We call a continuous linear functional a distribution or a generalized function.

**Remark 1.2.4.**

(1) For now the names generalized functions and distributions are synonymous as there is no difference for $\mathbb{R}$. We will discuss the difference in a later part of the manuscript, when it will be relevant.

(2) Warning! It might not be the case that $f|_U \in C^\infty_c(U)$ even if $f \in C^\infty_c(V)$ and $U \subset V$.

Recall that a function $f$ is locally $L^1$, denoted $f \in L^1_{\text{Loc}}$, if the restriction to any compact subset in its domain is an $L^1$ function. Given a real function $f \in L^1_{\text{Loc}}$ we define $\xi_f : C^\infty_c(\mathbb{R}) \to \mathbb{R}$ to be the generalized function $\xi_f(\phi) := \int_{-\infty}^\infty f(x) \cdot \phi(x)dx$.

Note that this integral converges as $\phi$ vanishes outside of $K$, and $f|_K, \phi|_K \in L^1$. These are sometimes called regular generalized functions.

**Exercise 1.2.5.** For any $f \in L^1_{\text{Loc}}$, show that $\xi_f$ is a well defined distribution.

The space of generalized functions on $\mathbb{R}$ is denoted by $C^{-\infty}(\mathbb{R}) := (C^\infty(\mathbb{R}))^\ast$. Note that we have $C(\mathbb{R}) \subset L^1_{\text{Loc}} \subset C^{-\infty}(\mathbb{R})$, where the second inclusion follows from the embedding $f \mapsto \xi_f$.

**Exercise 1.2.6.** Prove that there exists a function $f \in C^\infty_c(\mathbb{R})$ which is not the zero function. **Hint:** Use functions such as $e^{-1/(1-x)^2}$.

**Definition 1.2.7.** We say that a sequence of generalized functions $\{\xi_n\}_{n=1}^\infty$ weakly converges to $\xi \in C^{-\infty}(\mathbb{R})$ if for every $f \in C^\infty_c(\mathbb{R})$ we have

$$
\lim_{n \to \infty} \langle \xi_n, f \rangle = \langle \xi, f \rangle.
$$

Note that in particular this definition applies to functions, since as we have seen above they are contained in the space of generalized functions. Now we can give an equivalent definition of the space of generalized functions - as the completion of $C^\infty_c(\mathbb{R})$ with respect to the weak convergence. For this we need the notion of a weakly Cauchy sequence:
**Definition 1.2.8.** A sequence \( \{f_n\} \) is called a weakly Cauchy sequence if for every \( g \in C_c^\infty(\mathbb{R}) \) and \( \epsilon > 0 \) there exists a number \( N \in \mathbb{N} \) such that for all \( m, n > N \) we have that
\[
\left| \int_{-\infty}^{\infty} (f_n(x) - f_m(x))g(x)dx \right| < \epsilon.
\]

**Exercise 1.2.9.** Find a sequence of functions \( \{f_n\}_{n=1}^\infty \) converging weakly to \( f \) such that it does not converge point-wise to \( f \).

Note that weakly Cauchy sequences do not necessarily converge in \( C_c^\infty(\mathbb{R}) \). In particular, one can find a weakly Cauchy sequence that converges to the Dirac’s delta.

**Definition 1.2.10.** A sequence \( \phi_n \in C_c(\mathbb{R}) \) of continuous, non-negative, compactly supported functions is said to be an approximation of identity if:

1. \( \phi_n \) satisfy \( \int_{-\infty}^{\infty} \phi_n(x) \cdot dx = 1 \) (that is have total mass 1), and
2. for any fixed \( \varepsilon > 0 \), the functions \( \phi_n \) are supported on \( [-\varepsilon, \varepsilon] \) for \( n \) sufficiently large.

The reason for the name is that \( \delta_0 \) is the identity for the convolution operation that we will define later.

Such sequences can be generated, for example, by starting with a non-negative, continuous, compactly supported function \( \phi_1 \) of total mass 1, and by then setting \( \phi_n(x) = n\phi_1(nx) \).

**Exercise 1.2.11.** Find an approximation to identity.

Note that given \( \eta \in C^{-\infty}(\mathbb{R}) \) of the form \( \eta = \xi f \), we can recover the value of \( f \) at \( t \) via
\[
\lim_{n \to \infty} \langle \xi f, \phi_n(x + t) \rangle = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x)\phi_n(x + t)dx = f(t).
\]

**Exercise 1.2.12.** There is a natural isomorphism between the completion of \( C_c^\infty(\mathbb{R}) \) with respect to weak convergence and \( (C_c^\infty(\mathbb{R}))^* \) (as vector spaces).

1.3. Derivatives of generalized functions. Let \( f \in C_c^\infty(\mathbb{R}) \). Since \( \xi f'(\phi) = \int_{-\infty}^{\infty} f'(x) \cdot \phi(x)dx \) we can use integration by parts to deduce that
\[
\xi f'(\phi) = f(x) \cdot \phi(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \cdot \phi'(x)dx.
\]

However, since \( \phi \) and \( f \) has compact support, we know that \( f(x) \cdot \phi(x)|_{-\infty}^{\infty} = 0 \). Thus, we define \( \xi' f(\phi) := -\xi(\phi') \).

For example, the derivative of \( \delta_0 \) can be intuitively described as
\[
\delta_0'(x) := \begin{cases} 
\infty & x \to 0^- \\
-\infty & x \to 0^+ \\
0 & \text{otherwise.}
\end{cases}
\]
This description is incorrect, since we cannot evaluate generalized functions at specific points, and furthermore describing $\delta_0^\prime$, $\delta_0^\prime\prime$ in such a way is more complicated. In our definition above, when $\delta_0^{(n)}$ is applied to some $\phi \in C_c^\infty(\mathbb{R})$, we get

$$\delta_0^{(n)}(\phi) = (-1)^n \delta_0(\phi(n)) = (-1)^n \phi(n)(0).$$

**Exercise 1.3.1.** Find a function $F \in L^1_{\text{loc}}$ for which $F' = \delta_0$ as generalized functions.

### 1.4. The support of generalized functions

Let $U \subset \mathbb{R}$ be an open set and let $C_c^\infty(U)$ be the space of smooth functions $f : U \to \mathbb{R}$ supported in some compact subset of $U$. Given a compact subset $K$ of a Euclidean space $X$, we denote by $C_c^\infty(X)$ the space of smooth functions $f : X \to \mathbb{R}$ with $\text{supp}(f) \subseteq K$. In particular $C_c^\infty(K) \subseteq C_c^\infty(X)$ for every $K \subseteq X$.

We cannot evaluate a generalized function at a point. Therefore, we cannot just define its support as we did before by $\text{supp}(\xi) := \{x \in \mathbb{R} \mid \xi(x) \neq 0\}$. However, if for some neighborhood $U \subset \mathbb{R}$ we have for every $f \in C_c^\infty(U)$ that $\xi(f) = 0$, then evidently $\text{supp}(\xi) \subseteq U^c$. This leads us to the next definition:

**Definition 1.4.1.** For $\xi \in (C_c^\infty(\mathbb{R}))^*$ we say that $\xi|_U \equiv 0$ if $\langle \xi, f \rangle = 0$ for all $f \in C_c^\infty(U)$. Define $\text{supp}(\xi) = \bigcap_{\xi|_{D_\beta} \equiv 0} D_\beta$, where $D_\beta$ are taken to be closed.

For the definition to be well defined one needs to show:

**Exercise 1.4.2.** Let $U_1, U_2$ be open subsets of $\mathbb{R}$ and $\xi \in C^{-\infty}(\mathbb{R})$. Show that:

1. If $\xi|_{U_1} \equiv \xi|_{U_2} \equiv 0$ then $\xi|_{U_1 \cup U_2} \equiv 0$. **Hint:** Use partition of unity.
2. Show that if $I$ is a set of arbitrary cardinality and \{\(U_\alpha\)\}_{\alpha \in I}$ is a collection of open subsets of $\mathbb{R}$ with compact closures and $\xi|_{U_\alpha} \equiv 0$ for any $\alpha \in I$ then $\xi|_{\bigcup_{\alpha \in I} U_\alpha} \equiv 0$.

**Remark 1.4.3.** Note that $\text{supp}(\xi)$ is always a closed set.

**Example 1.4.4.** The support of $\delta_0$ is $\{0\}$.

**Remark 1.4.5.** Warning! While the support of $\delta_0^\prime$ is also $\{0\}$, given some $f \in C(\mathbb{R})$ for which $f(0) = 0$ but $f'(0) \neq 0$, we get that $\delta_0^\prime(f) = -\delta_0(f') = -f'(0) \neq 0$. In other words, having $f(0) = 0$ is not enough to get $\langle \delta_0^\prime, f \rangle = 0$, we need $f$ to vanish with all its derivatives, i.e. vanish in a neighborhood of 0.

**Exercise 1.4.6.** Let $\xi_1, \xi_2 \in C^{-\infty}(\mathbb{R})$ and $a, b \in \mathbb{R}$. Show that:

1. $\text{supp}(a\xi_1 + b\xi_2) \subseteq \text{supp}(\xi_1) \cup \text{supp}(\xi_2)$.
2. $\text{supp}(\xi) - \text{supp}(\xi)^c \subseteq \text{supp}(\xi') \subseteq \text{supp}(\xi)$.

**Exercise 1.4.7.** Show that all the generalized functions $\xi \in C_c^{-\infty}(\mathbb{R})$ which are supported on $\{0\}$ are of the form $\sum_{i=0}^n c_i \delta^{(i)}$ for some $n \in \mathbb{N}_0$ and $c_i \in \mathbb{R}$ in two steps.
Exercise 1.5.2. Show that for smooth function:

\[ \langle \xi x^k, f \rangle = \langle \xi, x^k f \rangle = 0 \]

for every \( f \in C_c^\infty(\mathbb{R}) \).

Exercise 1.5.3. If \( \xi x^k = 0 \) for some \( k \in \mathbb{N} \), then \( \xi = \sum_{i=0}^{k-1} c_i \delta_i \) for some \( c_i \in \mathbb{R} \).

1.5. Products and convolutions of generalized functions.

Definition 1.5.1. Let \( f \in C_c^\infty(\mathbb{R}) \) and \( \xi \in C_c^{-\infty}(\mathbb{R}) \). We would like to have

\[(f \cdot \xi)(\phi) = \int_{-\infty}^{\infty} \xi(x) \cdot f(x) \cdot \phi(x) dx.\]

Thus, we define \((f \cdot \xi)(\phi) := \langle f, \phi \rangle\).

While we can multiply every such function \( f \) and generalized function \( \xi \), the product of two generalized functions is not always defined. Notice that indeed in some topologies the product of two Cauchy sequences is not always a Cauchy sequence, so we might not be able to approximate the product of two generalized functions by the product of their approximations.

Recall that given two functions \( f, g \), their convolution is defined by \((f * g)(x) := \int_{-\infty}^{\infty} f(t) \cdot g(x-t) dt\). The convolution of two smooth functions is always smooth, if it exists. In addition, if \( f \) and \( g \) have compact support, then so does \( f * g \):

Exercise 1.5.2.

1. Show that \( \text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g) \), where \( \text{supp}(f) + \text{supp}(g) \) is the Minkowski sum of \( \text{supp}(f) \) and \( \text{supp}(g) \). Thus \( f, g \in C_c^\infty(\mathbb{R}) \) implies \( f * g \in C_c^\infty(\mathbb{R}) \).

2. Find an example in which the left hand side is strictly contained in the right hand side.

Given \( f, g \in C_c^\infty(\mathbb{R}) \) we can write \((f * g)(x) = \xi_f(\tilde{g}_x)\), where \( \tilde{g}_x(t) := g(x-t)\). This motivates us to define the convolution \( \xi * g \) as the function \((\xi * g)(x) = \xi(\tilde{g}_x)\). Note that the convolution of a smooth function and a generalized function is always a smooth function:

Exercise 1.5.3. Show that for \( \phi \in C_c^\infty(\mathbb{R}) \) and \( \xi \in C^{-\infty}(\mathbb{R}) \) we get that \( \xi \ast \phi \) is a smooth function.

Next is the definition for convolution of two generalized functions. This will not be defined for every pair of generalized functions, but for pairs such that at least one of the generalized functions have compact support. Firstly, for \( \xi_f, \xi_g \in C_c^{-\infty}(\mathbb{R}) \) we would like to have:

\[(\xi_f * \xi_g)(\phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot g(x-t) \cdot \phi(x) dt dx.\]

Interchanging the order of integration yields:

\[(\xi_f * \xi_g)(\phi) = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(x-t) \cdot \phi(x) dx dt.\]
When convolving functions, the arguments of the convolved functions sum up to the convolution’s argument (e.g., \((f \ast g)(x) := \int_{-\infty}^{\infty} f(t) \cdot g(x-t) \, dt\), and \(x = t + (x-t)\)).

In our case, we denote \(\hat{\phi}(x) := \phi(-x)\), and write:
\[
\int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(x-t) \cdot \hat{\phi}(-x) \, dx \, dt = \int_{-\infty}^{\infty} f(t) \cdot (\xi_g \ast \hat{\phi})(-t) \, dt = \xi_f(\xi_g \ast \phi).
\]

**Definition 1.5.4.** We define \(\langle \xi_f \ast \xi_g, \phi \rangle := \langle \xi_f, \xi_g \ast \phi \rangle\).

However, some formal justification is required. Given a compact \(K \subset \mathbb{R}\), we say \(\rho\) is a cutoff function of \(K\) if \(\rho|_K \equiv 1\) and \(\rho|_V \equiv 0\), where \(\mathbb{R} \setminus V\) has compact closure.

**Exercise 1.5.5.** Let \(K\) and \(V\) be as above. Show that there exists a continuous cutoff function. **Hint:** use Urysohn’s Lemma.

Thus, given some \(\xi \in C_c^{-\infty}(\mathbb{R})\) with \(\text{supp}(\xi) \subset K\) we have that \(\xi(\phi) = \xi(\rho_K \cdot \phi)\). This enables us to define \(\xi\) as a functional over all \(C^\infty(\mathbb{R})\) and not only on \(C_c^\infty(\mathbb{R})\).

For every \(\phi \in C^\infty(\mathbb{R})\) we define \(\xi(\phi) = \xi(\rho_K \cdot \phi)\) where \(K := \text{supp}(\xi) \subset \mathbb{R}\).

**Exercise 1.5.6.** Let \(\xi \in C^{-\infty}(\mathbb{R})\). In an exercise above we showed: if \(\phi \in C_c^\infty(\mathbb{R})\) then the convolution \(\xi \ast \phi\) is compact. Show that if \(\phi\) is smooth, and \(\text{supp}(\xi)\) is compact, then \(\xi \ast \phi\) is still smooth.

To summarize, the convolution of two compactly supported distributions is well defined and compactly supported, while the convolution of a compactly supported distribution with an arbitrary distribution is well defined, but usually not compactly supported.

**Exercise 1.5.7.** Show the following identities for any compactly supported distributions \(\xi_1, \xi_2\) and \(\xi_3\) in \(C^{-\infty}(\mathbb{R})\).

1. \(\delta_0 \ast \xi_1 = \xi_1\).
2. \(\delta' \ast \xi_1 = \xi_1'\).
3. \(\xi_1 \ast \xi_2 = \xi_2 \ast \xi_1\).
4. \(\xi_1 \ast (\xi_2 \ast \xi_3) = (\xi_1 \ast \xi_2) \ast \xi_3\).
5. \((\xi_1 \ast \xi_2)' = \xi_1 \ast \xi_2' = \xi_1' \ast \xi_2\).

**Exercise 1.5.8.** Let \(K \subset \mathbb{R}\) be a compact set. Construct a function \(f \in C_c^\infty(\mathbb{R})\) such that \(f|_K \equiv 1\) and \(f|_U \equiv 0\) for some neighborhood \(K \subset U\) (Hint: convolve a suitable approximation of identity with the indicator function of \(K\)).

1.6. **Generalized functions on \(\mathbb{R}^n\).** All the notions above make sense for functions and generalized functions in several variables. The definitions and the statements literally generalize to this case. For example, let us restate the definition of convergence in \(C^\infty_c(\mathbb{R}^n)\).
Definition 1.6.1 (Convergence in $C_c^\infty(\mathbb{R}^n)$). Given $f \in C_c^\infty(\mathbb{R})$ and a sequence $(f_n)_{n=1}^\infty$ of smooth functions with compact support, we say that $(f_n)_{n=1}^\infty$ converges to $f$ in $C_c^\infty(\mathbb{R}^n)$ if:

1. There exists a compact set $K \subset \mathbb{R}^n$ for which $\bigcup_{n \in \mathbb{N}} \text{supp}(f_n) \subseteq K$.

2. For every multi-index $\alpha$, the partial derivatives $(f_n^{(\alpha)})_{n=1}^\infty$ converge uniformly to the partial derivative $f^{(\alpha)}$.

1.7. Generalized functions and differential operators. A differential equation is given by the equality $Af = g$, where $A$ is a differential operator. Assume $A$ is a linear differential operator which is invariant under translations, i.e. we have that $AR_t(f) = R_t(Af)$, where $R_t(\phi)(x) = \phi(x + t)$ for some constant $t$.

An example of such operator is a differential operators with fixed coefficients, e.g. $Af := f'' + 5f' + 6f$.

A simple case is the equation $AG = \delta_0$. Given a solution $G$, using the invariance of $A$ under translations, we get that $AG_x = \delta_x$, for $G_x(t) := G(t - x)$. Using the exercise above we can show that $A(f * h) = (Af) * h$ for any two functions $f, h$ and then deduce that $A(G * g) = AG * g = \delta_0 * g = g$. Hence, we can find a general solution $f$ for $Af = g$ by solving a single simpler equation $AG = \delta_0$. The solution $G$ is called Green's function of the operator.

Exercise 1.7.1. Let $A$ be a differential operator with constant coefficients (i.e. as above).

1. Choose any solution for the equation $AG = \delta_0$, and describe the conditions $G$ has to meet without using generalized functions.

2. Without using generalized functions, explain the equation $A(G * g) = g$ we got for the solution $G$.

3. Solve the equation $\Delta f = \delta_0$ (where $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian).

1.8. Regularization of generalized functions.

Definition 1.8.1. Let $\{\xi_\lambda\}_{\lambda \in \mathbb{C}}$ be a family of generalized functions. We say the family is analytic if $\langle \xi_\lambda, f \rangle$ is analytic as a function of $\lambda \in \mathbb{C}$ for every $f \in C_c^\infty(\mathbb{R})$.

Example 1.8.2. We denote $x_\lambda^+ := \begin{cases} x^\lambda & x > 0 \\ 0 & x \leq 0 \end{cases}$.

and define the family by $\xi_\lambda := x_\lambda^+ \Re(\lambda) > -1$. The behavior of the function changes as $\lambda$ changes: When $\Re(\lambda) > 0$ we have a continuous function; if $\Re(\lambda) = 0$ we get a step function and for $\Re(\lambda) \in (-1, 0)$, $x_\lambda^+$ will not be bounded. We would like to extend the definition analytically to $\Re(\lambda) < -1$.

Deriving $x_\lambda^+$ (both as a complex function or as we defined for generalized function) gives $\xi_\lambda = \lambda \cdot \xi_{\lambda-1}$. This is a functional equation which enables us to define
\( \xi_{\lambda-1} := \xi_{\lambda}^{\prime} \lambda \), and thus extend \( \xi_{\lambda} \) to every \( \lambda \in \mathbb{C} \) such that \( \text{Re}(\lambda) > -2 \), and for every \( \lambda \) by reiterating this process. This extension is not analytic, but it is meromorphic: it has a pole in \( \lambda = 0 \), and by the extension formula, in \( \lambda = -1, -2, \ldots \).

This is an example of a meromorphic family of generalized functions. We now give a formal definition. The family \( \{\xi_{\lambda}\}_{\lambda \in \mathbb{C}} \) has a set of poles \( \{\lambda_{n}\} \) (poles are always discrete), whose respective orders are denoted \( \{d_{n}\} \). A family of generalized functions is called meromorphic if every pole \( \lambda_{i} \) has a neighborhood \( U_{i} \), such that \( \langle \xi_{\lambda}, f \rangle \) is analytic for every \( f \in C_{\infty}^{\infty}(\mathbb{R}) \) and \( \lambda_{i} \neq \lambda \in U_{i} \).

**Exercise 1.8.3.** Find the order and the leading coefficient of every pole of \( \xi_{\lambda} := x_{+}^{\lambda} \).

**Example 1.8.4.** For a given \( p \in \mathbb{C}[x_{1}, \ldots, x_{n}] \), similarly to before set,

\[
p_{+}(x_{1}, \ldots, x_{n})^{\lambda} := \begin{cases} 
    p(x_{1}, \ldots, x_{n})^{\lambda} & x > 0 \\
    0 & x \leq 0
\end{cases}.
\]

The problem of finding the meromorphic continuation of a general polynomial was open for some time. It was solved by J. Bernstein by proving that there exists a differential operator \( Dp_{+}^{\lambda} := b(\lambda) \cdot p_{+}^{\lambda-1} \), where \( b(\lambda) \) is a polynomial pointing on the location of the poles.

**Exercise 1.8.5.** Solve the problem of finding an analytic continuation for \( p_{+}(x_{1}, \ldots, x_{n})^{\lambda} \) in the following cases:

1. \( p(x, y, z) := x^{2} + y^{2} + z^{2} - a \) and \( a \in \mathbb{R} \).
2. \( p(x, y, z) := x^{2} + y^{2} - z^{2} \).
2. Lecture 2: Topological Properties of $C^\infty_\infty(\mathbb{R}^n)$

We want to analyze the space of distributions $C^{-\infty}(\mathbb{R}^n)$. For this aim, we want to introduce a topology on this space.

2.1. Normed space.

Definition 2.1.1. A normed space over $\mathbb{R}$ is a vector space $V$ over $\mathbb{R}$ with a function $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ satisfying

(i) $||\lambda v|| = |\lambda| \cdot ||v||$
(ii) $||v + w|| \leq ||v|| + ||w||$
(iii) $||v|| = 0 \iff v = 0$

If we weaken (iii) to state only $||0|| = 0$ we will get the definition of a semi-norm.

The norm defines a Hausdorff topology on $V$.

Example 2.1.2. (i) $l^p := \{\text{sequences } x_n \text{ in } \mathbb{R} | \sum |x_n|^p < \infty\}$
(ii) $L^p(\mathbb{R}) := \{\text{measurable } f : \mathbb{R} \to \mathbb{R} \mid |f|^p \text{ is integrable on } \mathbb{R}\}$.
(iii) $C^p(\mathbb{R}) := \{\text{functions with } p \text{ continuous bounded derivatives}, ||f|| := \sum_{i=1}^n \sup_{x \in \mathbb{R}} |f^{(i)}(x)|\}$. 

Let $V$ be a normed space, and $B := \{v \in V \mid ||v|| \leq 1\}$ be the unit ball.

Exercise 2.1.3. If dim $V$ is finite then $K$ is compact.

Corollary 2.1.4. Any finite-dimensional subspace of any normed space is closed.

Proof. Let $V$ be a normed space, $W \subset V$ be a finite-dimensional subspace. Let $v \in \overline{W}$, and let $K$ be the ball in $W$ with center at 0 and radius $2||v||$. Then $v$ lies in the closure of $K$. On the other hand, $K$ is compact and thus closed. Thus $v \in K \subset W$. \qed

Infinite-dimensional subspaces are not always closed. They might even be dense - for example the space of bounded infinitely-differentiable functions in the space of all bounded continuous functions. One can also have non-continuous linear functionals - these are precisely the non-zero functionals with dense kernels.

Proposition 2.1.5. If $B$ is compact then dim $V$ is finite.

Proof. Note that $B$ can be covered by open balls of radius 1/2: $B \subset \bigcup_{x \in B} B(x, 1/2)$. If $B$ is compact then this cover has a finite subcover. Denote the centers of the subcover by $\{x_i\}_{i=1}^n$, and let $W := \text{Span}(\{x_i\}_{i=1}^n)$. Then

$$B \subset \bigcup_{i=1}^n B(x_i, 1/2) \subset W + 1/2B \subset W + 1/4B \subset \ldots$$
Thus, any $v \in B$ can be presented as $w_k + z_k$ for any $k \in \mathbb{N}$, where $w_k \in W$ and $\|z_k\| < 2^{-k}$. Thus, $v = \lim w_k$. But $W$ is finite-dimensional, thus closed, and thus $v \in W$. Thus $V = W$ and thus $\dim V = n$. □

2.2. Topological vector spaces.

**Definition 2.2.1.** A topological vector space (or linear topological space) is a linear space with a topology such that multiplication by scalar and vectors addition are continuous. More precisely, there exist continuous operations:

1. $+: V \times V \to V$,
2. $\cdot: \mathbb{R} \times V \to V$.

This demand limits the topologies we can have on $V$.

**Remark 2.2.2.** In this definition $V$ is a vector space over $\mathbb{R}$, but in the same way one defines topological vector spaces over any topological field, e.g. over $\mathbb{C}$ or over the field of $p$-adic numbers that we will define later.

Since addition is continuous, so is translation by a constant vector. This makes all points of a topological vector space similar - the open neighborhoods of every point $x$ are, roughly speaking, the same as those of $0$. This property is called homogeneity.

We assume the topological vectors spaces we consider are well behaved. More specifically, we assume all topological vector spaces are Hausdorff, and locally convex (see definition bellow). Note that given a non-Hausdorff space $V$, we can quotient $V$ by the closure of $\{0\}$ and get a Hausdorff space.

**Definition 2.2.3.** Let $V$ be a topological vector space over $\mathbb{R}$.

1. We say that a set $A \subseteq V$ is convex if for every $a, b \in A$ the linear combination $ta + (1 - t)b \in A$ for any $t \in [0, 1]$.
2. We say that $V$ is locally convex if it has a basis of its topology which consists of convex sets.
3. For every open convex set $0 \in C$ in $V$ and $x \in V$ we define a semi-norm $N_C(x) = \inf\{\alpha \in \mathbb{R}_{\geq 0} : \frac{x}{\alpha} \in C\}$.
4. We say that a set $W \subseteq V$ is balanced if $\lambda W \subseteq W$ for all $\lambda \in \mathbb{R}$ satisfying $|\lambda| \leq 1$. Note that a convex set $C$ is balanced $\iff$ it is symmetric $(C = -C)$.

**Exercise 2.2.4.** Let $V$ be a topological vector space over $\mathbb{R}$.

1. Show that for every neighborhood $U$ of $0$ there exists an open balanced set $W$ such that $0 \in W \subseteq U$.
2. Find a topological vector space which is not locally convex (not necessarily of finite dimension).
3. Prove that $V$ is Hausdorff $\iff \{0\}$ is a closed set.
(4) Show that if $V$ is finite dimensional and Hausdorff it is isomorphic to $\mathbb{R}^n$.

**Remark 2.2.5.** From the homogeneity of $V$, we get that $\{0\}$ is a closed set $\iff$ $\{x\}$ is a closed set $\forall x \in V$. The previous exercise shows that a linear topological space satisfies the $T_1$ separation axiom $\iff$ it satisfies $T_2$.

**Exercise 2.2.6.** Let $0 \in C$ be an open convex set in a topological vector space $V$.

1. Show that $N_C(x) < \infty$ for all $x \in V$.
2. Show that if furthermore $C$ is balanced then $N_C(x)$ is a semi-norm (that is satisfies all the axioms of a norm, but can get zero values for non-zero input).

In a locally convex space we have a basis for the topology consisting of convex sets. We can assume all the sets are symmetric. Firstly, note that from the homogeneity of the space it is enough to show this for open sets around 0. Then, given any open convex neighborhood $A$ of 0, we know $A \cap -A$ is a (non-empty) symmetric convex open subset of $A$. We therefore have a basis for our topology consisting of symmetric convex sets.

Furthermore, there is a bijection between semi-norms on the space and symmetric convex sets. Given a semi-norm $N$ on $V$, the bijection maps $N$ to its unit ball $\{x \in V \mid N(x) \leq 1\}$ (exercise: see this is indeed symmetric and convex!). Note the semi-norm $N_C(x)$ we defined is not a norm. Indeed, if $C$ contains the subspace $\text{span}\{v\}$ for a non-zero $v$, we get that $N_C(v) = 0$ where $v \neq 0$. However, given the basis $T$ for our topology, we can not get $N_C(v) = 0$ for all sets $C \in T$ since in this case we would have $\text{span}\{v\} \subseteq \bigcap_{C \in T} C$, contradicting the Hausdorffness of our space.

**Definition 2.2.7.** A set $C \subseteq V$ is absorbent if $\forall x \in V$ there exists $\lambda \in \mathbb{R}$ such that $\frac{x}{\lambda} \in C$, i.e. multiplying $C$ by a big enough scalar can reach every point in the space. For absorbent $C \subseteq V$ we have that $N_C(v) < \infty$ for all $v \in V$. Note that every open set containing 0 is absorbent, and thus we can define a semi-norm for every set in the basis of the topology at $\{0\}$.

**Example 2.2.8.** The segment $\{(x,0) \mid x \in [0,1]\}$ in $\mathbb{R}^2$ is not absorbent, and for $y = (1,0)$ we get $n_C(y) = \infty$.

**Exercise 2.2.9.** Find a locally convex topological vector space $V$ such that $V$ has no continuous norm on it. That is, every convex open set $C$ contains a line $\text{span}\{v\}$, so $N_C(v) = 0$.

In conclusion, a locally convex space possesses a basis for its topology consisting of sets which define semi-norms. Some authors use this as the definition of a locally convex space.

Generalizing the proof of Proposition 2.1.5, one can prove the following theorem.
Theorem 2.2.10 ([Rud06, Theorem 1.22]). Every locally compact topological vector space has finite dimension.

2.3. Defining completeness. Given a metric space $X$, a point belongs to the closure of a given set $U$ if and only if it is the limit of a sequence of points in $U$. The convergence of the sequence $(a_n)_{n=1}^\infty$ to the point $x$ is defined by requiring that for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(a_n, x) < \epsilon$ whenever $n \geq N$. This is equivalent to requiring that for any neighborhood $U$ of $x$ there is some $N \in \mathbb{N}$ such that $a_n$ belongs to $U$ for all $n \geq N$.

For a general topological vector space $V$, even though we do not necessarily have a metric on $V$, we can define Cauchy sequence:

**Definition 2.3.1.** A sequence $(x_n)_{n=1}^\infty \subset V$ is called a Cauchy sequence, if for every neighborhood $U$ of $0 \in V$ there is $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_n - x_m \in U$.

**Remark 2.3.2.** More generally, if $X$ has a uniform topology, then we can define a notion of a Cauchy sequence. We will not give the definition of a uniform topology, but we note that any topological group possesses a uniform topology, and indeed one can define a notion of a left (resp. right) Cauchy sequence as follows: $(x_n)_{n=1}^\infty$ is a Cauchy sequence if for every neighborhood $U$ of $e \in G$ there is an index $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x^{-1}_m x_n \in U$ (resp. $x_n x^{-1}_m \in U$).

**Definition 2.3.3.** Let $V$ be a topological vector space.

1. $V$ is called sequentially complete if every Cauchy sequence in it converges.
2. A subset $Y \subseteq V$ is called sequentially closed if every Cauchy sequence $(y_n)_{n=1}^\infty$ in $Y$ converges to a point $y \in Y$.

The next example shows that we can have sets $Y$ that are sequentially complete but not closed. This example also shows that if the topology is too strong (e.g. not first countable), then the notion of Cauchy sequence might not be useful.

**Example 2.3.4.** Let $X$ be the real interval $[0, 1]$ and let $\tau$ be the co-countable topology on $X$; that is, $\tau$ consists of $X$ and $\emptyset$ together with all those subsets $U$ of $X$ whose complement $U^C$ is a countable set. Let $A = [0, 1)$, and consider its closure $\overline{A}$. We have that $\{1\}$ is not open because $X \setminus \{1\} = [0, 1)$ is not countable, and thus $\overline{A} = [0, 1]$. Since $1$ is not an element of $A$, it must be a limit point of $A$.

Suppose that $(a_n)_{n=1}^\infty$ is any sequence in $A$. Let $B = \{a_1, a_2, \ldots\}$ and let $U = B^c$ be its complement. Then $1 \in U$ and since $B$ is countable, it follows that $U$ is an open neighborhood of $1$ which contains no member of the sequence $(a_n)_{n=1}^\infty$. It follows that no sequence in $A$ can converge to the limit point $1$. This argument can be applied to show that $A$ has no Cauchy sequences, so it is (trivially) sequentially closed, but is not closed.
Definition 2.3.5. Let $V$ be a topological vector space.

(1) An embedding $i : V \hookrightarrow W$ is called a strict embedding if $i : V \rightarrow i(V)$ is an isomorphism of topological vector spaces.

(2) $V$ is called complete if for every strict embedding $\phi : V \hookrightarrow W$, the image $\phi(V)$ is closed.

Remark 2.3.6.

(1) Equivalently, we can define that a space $V$ is complete if every Cauchy net is convergent. From this definition it can be easily seen that any complete space $X$ is also sequentially complete.

(2) In the category of first countable topological vector spaces, completeness is equivalent to sequentially completeness, and indeed, there the notion of a Cauchy net is equivalent to the notion of a Cauchy sequence, and a set $Y \subseteq X$ is closed $\iff$ it is sequentially complete.

Exercise 2.3.7. Find a sequentially complete space which is not complete. Hint: see the above example.

Definition 2.3.8. Let $V$ be a topological vector space. A space $\tilde{V}$ is a completion of $V$ if $\tilde{V}$ is complete and there is a strict embedding $i : V \rightarrow \tilde{V}$ and $i(V)$ is dense in $\tilde{V}$.

Remark 2.3.9. We can also use a universal property in order to define the completion of $V$. A strict (?) embedding $i : V \rightarrow \tilde{V}$ is a completion of $V$ if:

(1) $\tilde{V}$ is complete.

(2) For every map $\psi : V \rightarrow W$ where $W$ is complete, there is a unique map $\phi_W : \tilde{V} \rightarrow W$, such that $\psi \equiv \phi_W \circ i$.

Exercise 2.3.10 (*). Show that these two definitions of completeness are equivalent.

It is often easier to show that a space is complete using the universal property. In this way we avoid dealing with Cauchy nets or filters. However, in order to show such completion exists one has to use these notions.

Exercise 2.3.11.

(1) (*) Show that every Hausdorff topological vector space has a completion.

(2) Show that in the category of first countable topological vector spaces both definitions of completion are equivalent to being sequentially complete.

2.4. Fréchet spaces. Reminder: A Banach space is a normed space, which is complete with respect to its norm. A Hilbert space is an inner product space which is complete with respect to its inner product.
Theorem 2.4.1. (Hahn-Banach) Let $V$ be a normed topological vector space, $W \subseteq V$ a linear subspace, and $C \in \mathbb{R}_{>0}$. Let $f : W \to \mathbb{R}$ be a linear functional such that $|f(x)| \leq C \cdot \|x\|$ for all $x \in W$. Then there exists $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}|_W = f$ and $|\tilde{f}(x)| \leq C \cdot \|x\|$ for all $x \in V$.

Exercise 2.4.2. Let $V$ be a locally convex topological vector space (i.e., not necessarily normed), and let $f : W \to \mathbb{R}$ be a continuous linear functional, where $W \subseteq V$ is a closed linear subspace of $V$. Show that $f$ can be extended to $V$.

Definition 2.4.3. The space of all continuous functionals on a topological vector space $V$ is called the dual space and denoted by $V^*$.

Exercise 2.4.4. Let $W \subseteq V$ be infinite-dimensional vector spaces. Show that any linear functional on $W$ can be extended to a linear functional on $V$. (There is no topology in this exercise).

Definition 2.4.5. A topological space $(X, \tau)$ is said to be metrizable if there exists a metric which induces the topology $\tau$ on $X$.

Remark 2.4.6. Every normed space is Hausdorff and locally convex, since there is a basis of its topology consisting of open balls, which are convex. We also know that every normed space is metric. However, metrizability does not force local convexity and vice versa.

Definition: A topological space $X$ is called a Fréchet space if it is a locally convex, complete space which is metrizable.

Exercise 2.4.7. Show that for a locally convex topological vector space $V$ the following three conditions are equivalent.

1. $V$ is metrizable.
2. $V$ is first countable (that is it has a countable basis of its topology at every point).
3. There is a countable collection of semi-norms $\{n_i\}_{i \in \mathbb{N}}$ that defines the basis of the topology of $V$, i.e., $U_{i,\epsilon} = \{x \in V | n_i(x) < \epsilon\}$ is a basis of the topology at 0.

Exercise 2.4.8. Let $V$ be a locally convex metrizable space. Prove that $V$ is complete (and consequently is a Fréchet space) $\iff$ it is sequentially complete.

Recall that the completion of a normed space $V$ with respect to its norm is the quotient space $\hat{V}$ of all Cauchy sequences in $X$ under the equivalence relation $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty} \iff \lim_{n \to \infty} \|x_n - y_n\| = 0$. In particular, $\hat{V}$ is a Banach space. Completing $V$ with respect to a semi-norm $N$ results in the elimination of all elements $\{x \in V | n(x) = 0\}$. The quotient space equipped with the induced norm on the quotient then yields a Banach space.
Example 2.4.9. Let \( V \) be the space of step functions on \( \mathbb{R} \), and consider the semi-norm \( \|f\|_1 := \int \|f(x)\|dx \). The completion of \( V \) with respect to \( \|\cdot\|_1 \) is isomorphic to the Banach space \( L^1(\mathbb{R}) \) (equipped with the norm on the quotient).

Let \( V \) be a Fréchet space, then we have a family of semi-norms \( \{n_i\}_{i \in \mathbb{N}} \) on \( V \). We can form a new system of ascending semi-norms by replacing \( n_i \) with \( \max\{n_j\} \). Let \( V_i \) be the completion of \( V \) with respect to \( n_i \).

If \( n_i \) and \( n_j \) were norms (and not just semi-norms), which satisfy \( \forall x \in V, n_i(x) \geq n_j(x) \), we would get a continuous inclusion \( V_i \hookrightarrow V_j \). A sequence of ascending norms \( n_1 \leq n_2 \leq \ldots \) thus gives rise to a descending chain of completions \( V_1 \leftarrow V_2 \leftarrow V_3 \leftarrow \ldots \). Our space \( V \) is then an inverse limit, \( V = \lim_{i \in \mathbb{N}} V_i \), which in this case has a very nice description: it is an intersection \( V = \bigcap_{i \in \mathbb{N}} V_i \) of the Banach spaces defined above.

If \( n_i \) and \( n_j \) are semi-norms, we get a continuous map \( V_i \rightarrow V_j \) as every converging sequence is mapped to a converging sequence which need not be injective. In this case \( V \) will be the inverse limit \( \lim_{i \in \mathbb{N}} V_i \) where the topology on \( V \) is generated by all the sets of the form \( \varphi_i^{-1}(U_i) \) where \( U_i \) is an open set in \( V_i \) and \( \varphi_i : V = \lim_{i \in \mathbb{N}} V_i \rightarrow V_i \) is the natural projection map which is part of the data of \( \lim_{i \in \mathbb{N}} V_i \).

Example 2.4.10. The following are examples of Fréchet spaces.

1. \( V := C^\infty(S^1) \) is a Fréchet space. Define the norms \( \{n_i\}_{i \in \mathbb{N}} \) by \( \|f\|_{n_i} := \max_{j \leq i} \sup_{x \in S^1} |f^{(j)}(x)| \). The completion with respect to \( n_k \) is \( V_k = C^k(S^1) \), the space of \( k \)-times differentiable functions. This family of norms satisfies \( \forall x \in V \) we have that \( n_j(x) \leq n_i(x) \) if \( j \leq i \), so by the argument above we indeed have \( C^\infty(S^1) = \bigcap_{k \in \mathbb{N}} C^k(S^1) \).

2. \( V = C^\infty(\mathbb{R}) \) is a Fréchet space. Define \( n_i \) by \( \|f\|_{n_i} := \max_{j \leq i} \sup_{x \in K_i} |f^{(j)}(x)| \) where \( K_i = [-i, i] \). Notice that this gives an ascending chain of seminorms so this defines a Fréchet space \( V = \lim_{i \in \mathbb{N}} V_i \). A similar argument shows that \( C^\infty(\mathbb{R}^n) \) is a Fréchet space, as well as \( C^\infty(M) \) for every smooth manifold \( M \). In these cases we take the supremum over all the possible directional derivatives.

3. Let \( K \) be a compact set and \( n \in \mathbb{N}_0 \), then \( C^\infty_K(\mathbb{R}^n) \) is a Fréchet space. Let \( k \in \mathbb{N}_0 \), \( C^k_K(\mathbb{R}^n) \) is a Banach space and in particular a Fréchet space. \( C^\infty_c(\mathbb{R}^n) \) is not Fréchet.

Definition 2.4.11. The direct limit of an ascending sequence of vector spaces is the space \( V_\infty := \bigcup_{n \in \mathbb{N}} V_n \). This is not a Fréchet space, but a locally convex topological vector space. A convex subset \( U \subseteq V_\infty \) is open \( \iff \ U \cap V_n \) is open in \( V_n \) for all \( n \).
Every space of the form $C^\infty(K)$ can be given the induced topology from $C^\infty(\mathbb{R})$. Taking the union of the ascending chain

$$C^\infty([-1, 1]) \subset C^\infty([-2, 2]) \subset \ldots$$

gives all smooth functions with compact support $C^\infty_c(\mathbb{R}) = \lim_{n \to \infty} C^\infty([-n, n])$ as a direct limit. However, this is not a Fréchet space - it is a direct limit and not an inverse limit. A basis of the topology of $C^\infty_c(\mathbb{R})$ at 0 is given by the sets:

$$U_{(\epsilon_n, k_n)} := \sum_{n \in \mathbb{N}} \{ f \in C^\infty(\mathbb{R}) | \text{supp}(f) \subseteq [-n, n] \text{ and } |f^{(k_n)}(x)| < \epsilon_n \} ,$$

where $\epsilon_n \in \mathbb{R}_{>0}$, $k_n \in \mathbb{N}_0$ and $\Sigma$ denotes Minkowski sum, that is $A + B := \{ a + b | a \in A, b \in B \}$.

Exercise 2.4.12. Show that a sequence $(f_n)_{n=1}^\infty$ in $C^\infty_c(\mathbb{R})$ converges to $f \in C^\infty_c(\mathbb{R})$ with respect to the topology defined above if and only if it converges as was defined in the first lecture (Definition 1.6.1), i.e.,

1. There exists a compact set $K \subseteq \mathbb{R}$ s.t. $\bigcup_{n=1}^\infty \text{supp}(f_n) \subseteq K$.
2. For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$.

Remark 2.4.13. Notice that the topology on $C^\infty_c(\mathbb{R})$ is complicated - it is a direct limit of an inverse limit of Banach spaces!

Exercise 2.4.14. Show that taking a convex hull instead of a Minkowski sum (i.e., defining $U_{(\epsilon_n, k_n)} := \text{conv}_{n \in \mathbb{N}} \{ f \in C^\infty(\mathbb{R}) | \text{supp}(f) \subseteq [-n, n], f^{(k_n)}(x) < \epsilon_n \}$) will result in the same topology. This shows that $C^\infty_c(\mathbb{R})$ is a locally convex topological vector space (Note that this follows since this is a direct limit of Fréchet spaces).

Finally, Fréchet spaces have several more nice properties:

- Every surjective map $\phi : V_1 \to V_2$ between Fréchet spaces is an open map (it is enough to demand that $V_2$ is Fréchet and $V_1$ is complete).
- In the previous item, defining $K := \ker \phi$, one can show that the quotient $V_1/K$ is a Fréchet space, and $\phi$ factors through $V_1/K$, that is $\phi : V_1 \to V_1/K \to V_2$, where the map $V_1/K \to V_2$ is an isomorphism.
- Every closed map $\phi : V_1 \to V_2$ between Fréchet spaces can be similarly decomposed. Firstly, by showing $\text{Im}(\phi)$ is a Fréchet space, and then by writing $V_1 \to \text{Im}(\phi) \to V_2$.

2.5. Sequence spaces. An important family of examples of Fréchet spaces are sequence spaces.

Example 2.5.1. The space $\ell^p$ is the space of all sequences $(x_n)_{n=1}^\infty$ with values in $\mathbb{R}$, such that $\sum_{n=1}^\infty |x_n|^p < \infty$. It is a Banach space, and for $p = 2$ it is a Hilbert space.
Let \( \text{SW}(\mathbb{N}) \) be the space of all the sequences which decay to zero faster than any polynomial, i.e. \( \forall n \in \mathbb{N}, \lim_{i \to \infty} x_i \cdot i^n = 0 \). A family of norms one can consider when analyzing these spaces is \( \| (x_i) \|_n = \sup_{i \in \mathbb{N}} \{|x_i \cdot i^n|\} \). It is not hard to see that with respect to these norms every Cauchy sequence converges. Define the topology on \( \text{SW}(\mathbb{N}) \) using the family of norms \( \| \cdot \|_n \), then \( \text{SW}(\mathbb{N}) \) is a Fréchet space. This is an example of a Fréchet space which is not a Banach space.

Remark 2.5.2. How can we see every Cauchy sequence converges? Why is it not a Banach space?

The dual space \( \text{SW}(\mathbb{N})^* \) is \( \{(x_i)_{i=1}^{\infty} \mid \exists n, c : |x_i| < c \cdot i^n \} \). This is a union of Banach spaces, as opposed to the intersection we had when defining the completion of a Fréchet space (we will discuss the dual space more thoroughly next lecture). Note that both \( \text{SW}(\mathbb{N}) \) and \( \text{SW}(\mathbb{N})^* \) contain the subspace of all sequences with compact support (that is sequences with finitely many non-zero elements).

Example 2.5.3. Smooth functions on the unit circle, \( C^\infty(S^1) \), correspond to sequences \( (x_i)_{i=1}^{\infty} \) decaying faster than any polynomial. More precisely, we can view \( f \in C^\infty(S^1) \) as a periodic function in \( C^\infty(\mathbb{R}) \) which can be written as \( f(x) = \sum_{n=-\infty}^{\infty} a_n \cdot e^{int} \) (for functions of period 1). We thus attach to \( f \) the sequence \( (a_n)_{n=1}^{\infty} \) where \( a_n \) decays faster than any polynomial.

Exercise 2.5.4.

1. Show that the Fourier series map \( \mathcal{F} : C^\infty(S^1) \to \text{SW}(\mathbb{Z}) \) via \( f \mapsto a_n \) is an isomorphism of Fréchet spaces. In other words, show that it is a bijection and that for any semi-norm \( P_i \) of \( \text{SW}(\mathbb{Z}) \) there exists a semi-norm \( S_j \) of \( C^\infty(S^1) \) and \( C \in \mathbb{R} \) such that for any \( f \in C^\infty(S^1) \) we have that \( \| \mathcal{F}(f) \|_{P_i} < C \cdot \| f \|_{S_j} \) (and vice-versa, or use Banach’s open mapping theorem for Fréchet spaces).

2. Define a Fréchet topology on \( S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) \mid \lim_{x \to \pm \infty} f^{(n)}(x) \cdot x^k \to 0 \text{ for all } k \} \).
3. Lecture 3: topology and filtrations on $C^{-\infty}(\mathbb{R}^n)$

3.1. Topologies on the space of distributions.

**Remark 3.1.1.** Let $U \subseteq \mathbb{R}^n$ be an open set, we define $C^{-\infty}(U)$ to be the continuous dual of $C^\infty_c(U)$.

**Definition 3.1.2.**

1. Let $V$ be a topological vector space. A subset $B \subseteq V$ is called bounded if for every open $U \subseteq V$ there exists $\lambda \in \mathbb{R}$ such that $B \subseteq \lambda \cdot U$. When the topology on $V$ is given by a sequence of norms, $B$ is bounded $\iff$ it is bounded with respect to every one of the norms.

2. Denote $V^* = \{ f : V \to \mathbb{R} : f \text{ is linear and continuous} \}$. There are many topologies one can define on $V^*$, we mention here two of those. Let $\epsilon > 0$ and $S \subseteq V$, and set $U_{\epsilon,S} = \{ f \in V^* : \forall x \in S, \ |f(x)| < \epsilon \}$.

   a. The weak topology on $V^*$, denoted $V^*_w$. The basis for the topology on $V^*_w$ at $0$ is given by:

   $$B_w := \{ U_{\epsilon,S} : \epsilon > 0, S \text{ is finite} \}.$$

   b. The strong topology on $V^*$, denoted $V^*_S$. The basis for the topology on $V^*_S$ at $0$ is given by:

   $$B_S := \{ U_{\epsilon,S} : \epsilon > 0, S \text{ is bounded} \}.$$

   In particular, every open set in $V^*_w$ is open in $V^*_S$.

By definition, a sequence $(f_n)_{n=1}^\infty \subseteq V^*$ converges to $f \in V^*$ if and only if for every $U_{\epsilon,S} \in B$ there exists $N \in \mathbb{N}$ s.t. $(f_n - f) \in U_{\epsilon,S}$ for $n > N$. That is, $\forall x \in S$ we have that $|f_n(x) - f(x)| < \epsilon$. Therefore $(f_n)_{n=1}^\infty$ converges to $f$ w.r.t the weak topology $\iff$ it converges point-wise, and it converges to $f$ w.r.t the strong topology $\iff$ it converges uniformly on every bounded set.

**Remark 3.1.3.** If the topology on $V$ is given by a collection of semi-norms (such as in Fréchet spaces), a set is bounded if and only if it is bounded with respect to every semi-norm.

**Theorem 3.1.4.** (Banach-Steinhaus) Let $V$ be a Fréchet space, $W$ be a normed vector space, and suppose there is a family $T_\alpha : V \to W$. If for all $v \in V$ we have that $\sup_{T \in F} \|T(v)\|_W < \infty$ then,

$$\sup_{T \in F, \|v\|_k = 1} \|T(v)\|_W < \infty.$$

**Example 3.1.5** (Fleeing bump function). Let $V = \mathbb{R}$ and let $\psi$ be a bump function. Notice that $g_n(x) = \psi(x+n)$ converges pointwise to 0 (and hence also in the weak topology). Note that $g_n$ does not converges uniformly to 0, but it does converges uniformly on bounded sets to 0, so it strongly converges to 0.
Assume $V$ is a Fréchet space. Recall that we can define $V$ as an inverse limit of Banach spaces $V = \bigcap_{i=1}^{\infty} V_i$ where $V_i$ is the completion of $V$ with respect to an increasing sequence of semi-norms $n_i$. If we dualize the system $\{V_i\}_{i=1}^{\infty}$, we get an increasing sequence $V_1^* \subseteq V_2^* \subseteq \ldots \subseteq V_S^* = \lim V_i^*$, and get that $V_S^*$ is a direct limit of Banach spaces (as a topological vector space).

**Exercise 3.1.6.** Let $S \subseteq C_c^\infty(\mathbb{R})$ be a bounded set, then there exists a compact $K \subseteq \mathbb{R}$ such that $S \subseteq C_K^\infty(\mathbb{R})$.

**Exercise 3.1.7.** Consider the embedding $C_c^\infty(\mathbb{R}) \hookrightarrow C^{-\infty}(\mathbb{R})$, defined by $f \mapsto \xi_f$. Show that:

1) This embedding is dense with respect to the weak topology on $C^{-\infty}(\mathbb{R})$.
2) This embedding is dense with respect to the strong topology on $C^{-\infty}(\mathbb{R})$.
3) $C^{-\infty}(\mathbb{R})$ with the weak topology is sequentially complete but not complete.
4) $C^{-\infty}(\mathbb{R})_w = C_c^\infty(\mathbb{R})^\#$ where the latter is the full dual space (that is all functionals, not necessarily continuous).
5) $C^{-\infty}(\mathbb{R})_S$ is complete.

### 3.2. Sheaf of distributions.

**Definition 3.2.1.** Let $U_1 \subseteq U_2 \subseteq \mathbb{R}^n$ be open sets. Every function $f \in C_c^\infty(U_1)$ can be extended to a function $\text{ext}_0 f \in C_c^\infty(U_2)$ by defining $\text{ext}_0 f \mid_{U_2 \setminus U_1} \equiv 0$, hence we have an embedding $C_c^\infty(U_1) \hookrightarrow C_c^\infty(U_2)$. This embedding defines a restriction map $C^{-\infty}(U_2) \to C^{-\infty}(U_1)$, mapping $\xi \mapsto \xi \mid_{U_1}$, with $(\xi \mid_{U_1}, f) = (\xi, \text{ext}_0 f)$.

**Remark 3.2.2.**

1) For an open $U \subseteq \mathbb{R}^n$, the topology we defined on $C_c^\infty(U)$ is generally not the same as the induced topology when considering it as a subspace of $C_c^\infty(\mathbb{R}^n)$.
2) For every compact $K \subset U$, we have $C_K^\infty(U) \subset C_c^\infty(U)$. Here the topology on $C_K^\infty(U)$ is indeed the induced topology from $C_c^\infty(U)$.

Next we prove that with respect to the restriction operation for distributions defined above, the space of distributions is equipped with a natural structure of a sheaf.

**Lemma 3.2.3** (Partition of unity). Let $f \in C_c^\infty(U)$, $I$ be an indexing set and $U = \bigcup_{i \in I} U_i$ be a union of open sets. Then $f$ can be written as a sum $f = \sum_{i \in I} f_i$ where $f_i \in C_c^\infty(U_i)$ and for every $x \in U$, the number $|\{i \in I : f_i(x) \neq 0\}|$ is finite.

**Proof.** We can assume that $U_i$ are open balls (otherwise, replace each $U_i$ by a collection of balls covering it). Set $K := \text{supp} (f)$, it is a compact set covered by $\bigcup_{i \in I} U_i$, so there exists a finite sub-cover: $K \subseteq \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} B(x_i, r_i)$. Since the cover is open and $K$ is closed, there exists $\epsilon > 0$ such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, r_i - \epsilon)$. 
Let $\rho_i$ be smooth non-negative bump functions satisfying $\rho_i|_{B(x_i, r_i)} \equiv 1$ and $\rho_i|_{B(x_i, r_i)^c} \equiv 0$. Since $\forall x \in K$ we have that $\sum_{i=1}^n \rho_i(x) \neq 0$, we can define:

$$ f_i = \begin{cases} \frac{\rho_i f}{\sum_{j=1}^n \rho_j} & x \in K \\ 0 & x \notin K \end{cases}. $$

This finishes the proof. \(\square\)

**Theorem 3.2.4.** With respect to the restriction map defined above, distributions form a sheaf, that is given an open $U \subseteq \mathbb{R}^n$, and open cover $U = \bigcup_{i \in I} U_i$, we have:

1. **(Identity axiom)** Let $\xi \in C^{-\infty}(U)$. If for every $i$ we have that $\xi|_{U_i} \equiv 0$, then $\xi|_U \equiv 0$.

2. **(Glueability axiom)** Given a collection of distributions $\{\xi_i\}_{i \in I}$, where $\xi_i \in C^{-\infty}(U_i)$, that agree on intersections, i.e. $\forall i, j \in I$ we have that $\xi_i|_{U_i \cap U_j} \equiv \xi_j|_{U_i \cap U_j}$, there exists $\xi \in C^{-\infty}(U)$ satisfying $\xi|_{U_i} \equiv \xi_i$ for any $i$.

**Proof.**

1. Given $f \in C_c^\infty(U)$ we need to show $\langle \xi, f \rangle = 0$. Indeed, by moving to a finite cover using the compactness of $\text{supp}(f)$ and using partition of unity we get a finite sum, $f = \sum_{i=1}^n f_i$, with $f_i \in C_c^\infty(U_i)$. Hence

$$ \langle \xi, f \rangle = \langle \xi, \sum_{i=1}^n f_i \rangle = \sum_{i=1}^n \langle \xi, f_i \rangle = 0. $$

2. We first use the fact that there exists a partition of unity (for non-compact sets). This means there exists a collection of functions $\{\lambda_i\}_{i \in I}$ such that $\sum_{i \in I} \lambda_i(x) = 1$ where $\text{supp}(\lambda_i) \subseteq U_i$, and the sum is finite for any $x \in U$. For any compact $K \subseteq U$ we then have that $\lambda_i|_K \equiv 0$ for all but finitely many $i$. Now fix some partition of unity $\{\lambda_i\}_{i \in I}$ and suppose we are given $\xi_i \in C^{-\infty}(U_i)$ which agree on pairwise intersections. Define $\langle \xi, f \rangle := \sum_{i \in I} \langle \xi_i, \lambda_i f \rangle$. Since $f$ is supported on some compact $K$ this sum is finite. It is clear that $\xi$ is linear, we need to prove that it is continuous, and that $\xi|_{U_i} = \xi_i$.

Let $f_n$ converge to $f$, where all functions lie in $C_c^\infty(U)$. Then also $\lambda_i \cdot f_n \to \lambda_i \cdot f$ as the multiplication $(f, g) \mapsto f \cdot g$ is continuous. Since $\bigcup_{n=1}^\infty \text{supp} f_n \cup \text{supp} f \subseteq K$ for some $K \subseteq U$, we have that $f \lambda_i \equiv 0$ for all but finitely many indices $i$ so we can write $\langle \xi, f \rangle := \sum_{i=1}^n \langle \xi_i, \lambda_i f \rangle$ and $\langle \xi, f_n \rangle := \sum_{i=1}^n \langle \xi_i, \lambda_i f_n \rangle$ for any $n$. By continuity of $\xi_i$ we get that $\langle \xi_i, \lambda_i \cdot f_n \rangle \to \langle \xi_i, \lambda_i \cdot f \rangle$ and therefore

$$ \langle \xi, f_n \rangle = \sum_i \langle \xi_i, \lambda_i \cdot f_n \rangle \to \sum_i \langle \xi_i, \lambda_i \cdot f \rangle = \langle \xi, f \rangle, $$
so $\xi$ continuous. Now let $f \in C_\infty^c(U_j)$, then

$$\langle \xi, f \rangle = \sum_i \langle \xi_i, \lambda_i f \rangle = \sum_i \langle \xi_j, \lambda_i f \rangle = \langle \xi_j, \sum_i \lambda_i f \rangle = \langle \xi_j, f \rangle,$$

where the second equality follows from the fact that $\lambda_i f \in C_\infty^c(U_j \cap U_i)$ and $\xi_i|_{U_j \cap U_i} \equiv \xi_j|_{U_j \cap U_i}$.

A second way to prove continuity of $\xi$ is working with the open sets in the topology of $C_\infty^c(U)$. As $\xi_i$ are continuous, they are bounded in some convex open set $0 \in B_i$, so $|\langle \xi_i, f \rangle| < \epsilon$ for every $f \in B_i$. Notice that $\text{conv}(\bigcup_{i \in I} B_i)$ is open in $\bigoplus_{i \in I} C_\infty^c(U_i)$, where each $B_i$ is an open set in $C_\infty^c(U_i)$ and hence a set in $\bigoplus_{i \in I} C_\infty^c(U_i)$ as $\text{conv}(\bigcup_{i \in I} B_i) \cap C_\infty^c(U_i) = B_i$. Notice that $\varphi(\text{conv}(\bigcup_{i \in I} B_i))$ is open. Now let $f \in \varphi(\text{conv}(\bigcup_{i \in I} B_i))$. We can write $f = \sum_{j=1}^n a_j f_i$ where $f_i \in B_{j_i}$ and $\sum a_i = 0$. Therefore $\xi(f) := \sum_j \xi_i(a_j f_i) < \sum a_i \cdot \epsilon = \epsilon$ and $\xi$ is bounded on $B$.

\[ \square \]

### 3.3. Filtration on spaces of distributions.

**Exercise 3.3.1.** Let $U \subset \mathbb{R}^n$, show that in $C_\infty^c(\mathbb{R}^n)$ we have that $\overline{C_\infty^c(U)} = \{ f \in C_\infty^c(\mathbb{R}^n) : \forall x \notin U, \forall$ differential operator $L, Lf(x) = 0 \}$.

Consider $U = \mathbb{R}^n \setminus \mathbb{R}^k$. We wish to describe the space of distributions supported on $\mathbb{R}^k$, which we denote $C_\infty^c(\mathbb{R}^n)$. Notice that:

$$C_\infty^c(\mathbb{R}^n) = \{ \xi \in C^{-\infty}(\mathbb{R}^n) | \forall f \in C_\infty^c(\mathbb{R}^n \setminus \mathbb{R}^k) \text{ we have } \langle \xi, f \rangle = 0 \}$$

and by continuity this is the same as:

$$\{ \xi \in C^{-\infty}(\mathbb{R}^n) | \forall f \in C_\infty^c(\mathbb{R}^n \setminus \mathbb{R}^k) \text{ we have } \langle \xi, f \rangle = 0 \} = \{ \xi \in C^{-\infty}(\mathbb{R}^n) | \xi|_V = 0 \},$$

where $V = \overline{C_\infty^c(U)}$ as described in Exercise 3.3.1. Notice that we can define a natural descending filtration on $V$ by:

$$V \subseteq V_m = \{ f \in C_\infty^c(\mathbb{R}^n) | \forall i \in \mathbb{N}_0^{n-k} \text{ where } |i| \leq m \text{ we have } \frac{\partial^i f}{\partial x^i}|_{\mathbb{R}^k} = 0 \}.$$

We immediately see that $f \in V_m$ implies $f \in V_{m-1}$, hence this is a descending chain. After dualizing, this defines an ascending filtration on $C_\infty^c(\mathbb{R}^n)$ by:

$$F_m(C^{-\infty}_\infty(\mathbb{R}^n)) = V_m^* = \{ \xi \in C^{-\infty}_\infty(\mathbb{R}^n) : \xi|_{V_m} = 0 \} \subseteq C^{-\infty}_\infty(\mathbb{R}^n).$$

**Exercise 3.3.2.** Show the following.

1. $\bigcap_{m \geq 0} V_m = V = \overline{C_\infty^c(\mathbb{R}^n \setminus \mathbb{R}^k)}$.
2. $\bigcup_{m \geq 0} F_m \neq C^{-\infty}_\infty(\mathbb{R}^n)$.  

(3) Let $U \subseteq \mathbb{R}^n$ be open and $\overline{U}$ compact. Show that for every $\xi \in C_0^\infty(\mathbb{R}^n)^*$ there exists $\xi' \in F_m$ such that $\xi|_U = \xi'|_U$, thus $\bigcup_{i=0}^\infty F_i$ covers $C_0^\infty(\mathbb{R}^n)^*$ locally.

(4) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function that fixes $\mathbb{R}^k$. Show that changing coordinates using $\varphi$ preserves $F_i$, that is for every $\xi \in F_i$ we have that $\varphi^*(\xi)$ via $\langle \varphi^*(\xi), f \rangle = \langle \xi, f \circ \varphi \rangle$ is in $F_i$.

**Theorem 3.3.3.** As vector spaces we have $F_m \simeq \bigoplus_{|i| \leq m} \frac{\partial^i(C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$ where $i \in \mathbb{N}_0^{n-k}$ and $\frac{\partial^i(C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$ is the image of $C^{-\infty}(\mathbb{R}^k)$ under the differential operator $\frac{\partial}{\partial x^i}$ (note that we only differentiate with respect to coordinates not lying in $\mathbb{R}^k$).

**Proof.** We prove here the statement for $m = 0$, and return to the case where $m > 0$ on Lecture 5. Define a map $\text{res}^* : C^{-\infty}(\mathbb{R}^k) \to F_0$ by $\langle \text{res}^*\xi, f \rangle = \langle \xi, f|_{\mathbb{R}^k} \rangle$ for every $\xi \in C^{-\infty}(\mathbb{R}^k)$. Notice that $\text{res}^*\xi(f) = 0$ for any $f \in F_0$ by definition so it is well defined.

Furthermore, $\text{res}^*$ is injective since if $\langle \text{res}^*\xi, f \rangle = \langle \xi, f|_{\mathbb{R}^k} \rangle = 0$ for all $f \in C_0^\infty(\mathbb{R}^n)$ then $\xi = 0$ since the restriction $\text{res} : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^k)$ is surjective.

It is left to prove surjectivity. Define an extension map $\text{ext}^* : F_0 \to C^{-\infty}(\mathbb{R}^k)$ by $\langle \text{ext}^*\eta, f \rangle = \langle \eta, \text{ext}(f) \rangle$ where $\text{ext}(f)|_{\mathbb{R}^k} = f$ for every $f \in C_0^\infty(\mathbb{R}^k)$. Note that this is well defined since if we choose a different extension $\text{ext}'(f)$ we get that $\langle \eta, \text{ext}'(f) - \text{ext}(f) \rangle = 0$ since $(\text{ext}'(f) - \text{ext}(f))|_{\mathbb{R}^k} \equiv 0$ and thus $\text{ext}^*\eta = \text{ext}'^*\eta$. Also, we have that $\text{ext}^*\eta$ is a continuous functional, since we can choose the extension $\text{ext}(f)$ in such a way that if $(f_n)_{n=1}^\infty$ converges to $f$ then $(\text{ext}(f_n))_{n=1}^\infty$ converges to $\text{ext}(f)$. Finally, note that since $\text{res}^*\text{ext}^*(\eta) = \eta$, we have that $\text{ext}^*$ is indeed surjective, and we are done.

**Remark 3.3.4.** Note that if we now define $G_m = \bigoplus_{|i| \leq m} \frac{\partial^i(C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$ we get that $G_m \simeq F_m/F_{m-1}$.

**Exercise 3.3.5.** Show that $G_m$ and $G_\{i\} = \frac{\partial^i(C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$, where $i$ is some multi-index, are not invariant under changes of coordinates, that is we might have that $\varphi(G_m) \neq G_m$ and $\varphi(G_\{i\}) \neq G_\{i\}$ for a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$. 
One motivation to define the $p$-adic numbers comes from number theory. Assume we are given a polynomial equation $p(x) = 0$ where $p \in \mathbb{Z}[x]$. If it has an integral solution $x_0 \in \mathbb{Z}$, then surely it satisfies the equation $p(x) = 0 \mod n$ for every $n \in \mathbb{N}$. Now, consider the converse question - if we know that it has a solution modulo $n$ for every $n \in \mathbb{N}$, does it have an integral solution in characteristic zero? In some cases, such as for quadratic forms, the answer, together with demanding that there also exists a real solution, is yes (see the Hasse principle for more on this). To know whether there exists a real solution, we can use simple methods from analysis. The question of whether an equation has a solution mod $n$ for every $n \in \mathbb{N}$ can be simplified in two steps. Firstly, by the Chinese remainder theorem it is enough to check whether the equation has a solution mod $p^n$ for every $n \in \mathbb{N}$. The second step is then by defining a ring $\mathbb{Z}_p$, called the ring of $p$-adic integers, such that if there exists a solution $x \in \mathbb{Z}_p$ it implies that there is a solution mod $p^n$ for every $n \in \mathbb{N}$. The field $\mathbb{Q}$ of $p$-adic numbers is then defined to be the field of fractions of $\mathbb{Z}_p$.

A different motivation for introducing the $p$-adic numbers comes from a more analytic point of view. One construction of the real numbers is via completing $\mathbb{Q}$ with respect to its absolute value. An interesting question is whether this can be generalized, that is what are the possible absolute value-like functions on $\mathbb{Q}$ and their completions. It turns out that besides the standard and the trivial absolute values, every absolute value (up to equivalence) is a $p$-adic absolute value (this is essentially Theorem 4.1.8 below). The $p$-adic numbers are then obtained as the completion of $\mathbb{Q}$ with respect to such an absolute value.

In this manuscript we take the second approach, starting with defining what properties we demand from an absolute value function.

### 4.1. Defining $p$-adic numbers.

**Definition 4.1.1.** A topological field is a field $F$, together with a topology, such that addition, multiplication and the multiplicative and additive inverses are continuous operations with respect to this topology.

**Definition 4.1.2.** Given a field $F$, an absolute value is a function $|\cdot| : F \to \mathbb{R}_{\geq 0}$ that satisfies:

1. The triangle inequality: $|x + y| \leq |x| + |y|$.
2. $|xy| = |x||y|$.
3. $|x| = 0 \Leftrightarrow x = 0$.

If furthermore $|x + y| \leq \max\{|x|, |y|\}$, we say that $|\cdot|$ is a non-Archimedean absolute value (and Archimedean otherwise).
For topological fields we demand the absolute value to be a continuous map. Notice that every absolute value satisfies $|1| = 1$ (as $|1| = |1| \cdot |1|$, and $|1| \neq 0$).

**Example 4.1.3.** The following are absolute values:

1. The trivial absolute value, defined by $|x|_0 := \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$

2. The standard absolute value on $\mathbb{R}$: $|\cdot|_{\infty} = \begin{cases} x & x > 0 \\ -x & x \leq 0. \end{cases}$

**Definition 4.1.4.** Let $p$ be a prime number. We define the $p$-adic absolute value of $x \in \mathbb{Q}$ by

$$|x|_p = \begin{cases} p^{-n}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases}$$

where $x = p^n \frac{a}{b}$ and $a, b \in \mathbb{Z}$, are coprime to $p$.

**Exercise 4.1.5.** Show that $|\cdot|_p$ is indeed an absolute value on $\mathbb{Q}$, and that it is non-Archimedean.

**Definition 4.1.6.** Two absolute values $|\cdot|$ and $|\cdot|'$ on $F$ are called equivalent, and denoted $|\cdot| \sim |\cdot|'$ if they induce the same topology on $F$.

**Exercise 4.1.7.** Let $|\cdot|$ and $|\cdot|'$ be two absolute values on a field $F$. Show that the following are equivalent:

1. $|\cdot|$ and $|\cdot|'$ are equivalent.

2. There exists $\alpha \in \mathbb{R}_{>0}$ such that $|\cdot| = (|\cdot|')^\alpha$.

3. Every sequence which is Cauchy with respect to $|\cdot|$ is Cauchy with respect to $|\cdot|'$.

**Theorem 4.1.8.** [Ostrowski’s Theorem] Every absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent to either $|\cdot|_p$ for a prime $p$, the standard absolute value $|\cdot|_{\infty}$ on $\mathbb{Q}$ induced from $\mathbb{R}$, or the trivial absolute value $|\cdot|_0$.

**Proof.** Let $|\cdot|$ be an absolute value, we show it must be one of the above by cases. Assume $|\cdot|$ is non-Archimedean, i.e. $|x + y| \leq \max\{|x|, |y|\}$, and set $a = \{x \in \mathbb{Z} : |x| < 1\}$. This set is non empty as $|0| = 0$, and since we assume $|\cdot|$ is non-Archimedean it is an ideal of $\mathbb{Z}$ since,

$$|x + \ldots + x| \leq |x|,$$

and thus if $x \in a$, meaning that $|x| < 1$, then $xy = \underbrace{x + \ldots + x}_{y \text{ times}} \in a$. Consider a prime $p$. If $|p| = 1$ for every prime, we get that $|x| = 1$ for every $0 \neq x \in \mathbb{Q}$, as $\frac{|x|}{|p|} = |p|^{-1}$, and thus $|\cdot|$ is the discrete absolute value. Thus we can assume that there exists $p \in a$ (note that for every integer $|m| \leq 1$ by (1)), implying that...
\[ p\mathbb{Z} \subseteq a \subseteq \mathbb{Z} \] and consequently \( p\mathbb{Z} = a \). Now, if we put \( s = -\frac{\log |a|}{\log p} \), we see that \( |p| = p^{-s} \). Taking \( x = p^n \frac{a}{b} \) where \( a, b, n \in \mathbb{Z} \) and \( a \) and \( b \) are coprime to \( p \) we get:

\[
|x| = |p^n \frac{a}{b}| = |p^n| \left| \frac{a}{b} \right| = |p|^n = p^{-ns} = |x|^s,
\]

showing \( |\cdot| \) is equivalent to \( |\cdot|^p \).

Now, assume \( |\cdot| \) is an Archimedean absolute value. We must have that \( |n| \geq 1 \) for all non-zero integers \( n \in \mathbb{Z} \). Otherwise, let \( n \) be the smallest positive number such that \( |n| < 1 \), and for every \( n < x \in \mathbb{N} \) write it in base \( n \):

\[
x = a_0 + a_1 n + a_2 n^2 + \ldots + a_r n^r, \quad 0 \leq a_i \leq n - 1, \quad n^r \leq x.
\]

We have \( |a_i| \leq a_i \leq n \), thus

\[
|x| = \sum_{i=0}^{r} |a_i| n^i \leq \sum_{i=0}^{r} n|n|^i = n \frac{(1 - |n|^{r+1})}{1 - |n|} \leq \frac{n}{1 - |n|}.
\]

Since \( n \frac{1}{1-|n|} \) is independent of \( r \), it bounds every \( x > n \), and thus we must have that \( |x| \leq 1 \) for every \( x > n \) as otherwise \( |x|^k > n \frac{1}{1-|n|} \) for \( k \) large enough. We get that \( |x| \leq 1 \) for \( x < n \) in the same way by considering \( n < x^k \) for \( k \) large enough. But this means that \( |x| \leq 1 \) for all \( x \in \mathbb{Z} \), so by the previous step it is equivalent to a \( p \)-adic absolute value, in contradiction to the fact that \( |\cdot| \) is Archimedean.

We can thus assume \( |n| \geq 1 \) for all \( n \in \mathbb{N} \). Recall the number \( r \) defined in (2) and note that \( r \leq \frac{\log x}{\log n} \). We now have:

\[
|x| \leq \sum_{i=0}^{r} |a_i| |n|^i \leq \left( 1 + \frac{\log x}{\log n} \right) n |n|^\frac{\log x}{\log n}.
\]

Using these bounds for \( x^k \):

\[
|x|^k \leq \left( 1 + \frac{k \log x}{\log n} \right) n |n|^\frac{k \log x}{\log n},
\]

implying

\[
|x|^\frac{1}{k} \leq \sum_{i=0}^{r} |a_i| |n|^i \leq \sqrt[k]{\left( 1 + \frac{k \log x}{\log n} \right) n |n|^\frac{k \log x}{\log n}}.
\]

By taking \( k \to \infty \), we get \( |x|^\frac{1}{k} \leq |n|^\frac{1}{\log n} \). But by interchanging \( x \) and \( n \) we can get that \( |n|^\frac{1}{\log n} \leq |x|^\frac{1}{\log x} \) (note that if \( n < x \) we can repeat this process for \( x \) and \( n^k \) for \( k \) large enough). Thus \( |x|^\frac{1}{\log x} = |n|^\frac{1}{\log n} = e^{\frac{\log |x|}{\log n}} \) is constant, implying that \( s = \frac{\log |x|}{\log x} = \frac{\log |n|}{\log n} \) is constant. Now, note that \( |x| = x^s \) for every \( x \) and get that,

\[
|x| = x^s = |x|^s,
\]

finishing the proof. \( \square \)
Exercise 4.1.9. Show that given a field $F$ and an absolute value $|\cdot|$ on it the topology it defines makes $F$ a topological field, i.e. that addition, multiplication, and the inverse operations are continuous.

Exercise 4.1.10. Non-Archimedean locally compact fields such as $\mathbb{Q}_p$ have some interesting properties. Prove the following two:

1. For every open ball $B(x,r) = \{y \in \mathbb{Q}_p^n : |x_i - y_i| < r\}$ of radius $r$ with center $x$ we have that $B(x,r) = B(x,r)$ for every $x' \in B(x,r)$.

2. Any two $p$-adic balls $B(x,r)$ and $B(x',r')$ in $\mathbb{Q}_p^n$ are either distinct or one contains the other.

We can now define the $p$-adic numbers.

Definition 4.1.11. Let $p$ be a prime number. We define the field of $p$-adic numbers $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$.

Remark 4.1.12.

1. The completion is defined just as we did in the case of the Archimedean norm on $\mathbb{Q}$; by equivalence classes of Cauchy sequences. Therefore, any element $a \in \mathbb{Q}_p$ is represented by a Cauchy sequence $\{a_n\}_{n=1}^\infty \subset \mathbb{Q}$ with respect to $|\cdot|_p$.

2. We get a space which is an uncountable field of characteristic 0, not algebraically closed, locally compact (every point has a compact neighborhood) and totally disconnected, i.e. every connected component is a point.

Definition 4.1.13. We define the $p$-adic integers $\mathbb{Z}_p$ to be the unit disc in $\mathbb{Q}_p$, explicitly $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

Exercise 4.1.14. Show that the $p$-adic absolute value extends from $\mathbb{Q}$ to $\mathbb{Q}_p$, that is show that for every Cauchy sequence $\{a_n\}_{n=1}^\infty$ of elements in $\mathbb{Q}$ the limit $\lim_{n \to \infty} |a_n|_p$ exists.

Remark 4.1.15. Notice that just like $\mathbb{R}$, this completion is not algebraically closed. Try to find an equation in $\mathbb{Q}_p$ for some $p$ which does not have a solution $\mathbb{Q}_p$.

4.2. Misc. -not sure what to do with them (add to an appendix about $p$-adic numbers?)

Theorem 4.2.1 (Taken from Koebliitz Theorem 2 page 11). Every equivalence class $a \in \mathbb{Q}_p$ for which $|a|_p \leq 1$ has exactly one representative Cauchy sequence of the form $\{a_i\}_{i=1}^\infty$ for which:

1) $0 \leq a_i < p^i$ for $i = 1, 2, \ldots$

2) $a_i \equiv a_{i+1}(\text{mod}(p^i))$ for $i = 1, 2, \ldots$

For the proof we will use the following lemma:
Lemma 4.2.2 (Taken from Koeblitz page 12). If \( x \in \mathbb{Q} \) and \( \|x\|_p \leq 1 \), then for any \( i \) there exists an integer \( \alpha \in \mathbb{Z} \) such that \( \|\alpha - x\|_p \leq p^{-i} \). The integer \( \alpha \) can be chosen in the set \( \{0, 1, 2, \ldots, p^i - 1\} \).

Proof. Let \( x = a/b \) written in the form where \( \gcd(a, b) = 1 \). Since \( \|x\|_p \leq 1 \) it follows that \( p \) does not divide \( b \) and therefore \( b \) and \( p^i \) are relatively prime. Then we can find \( m, n \in \mathbb{Z} \) such that \( bm + np^i = 1 \). The intuition is that \( bm \) is close to 1 up to a small \( p \)-adic length so it is a good approximation to 1 so \( am \) is a good approximation to \( a/b \). So we pick \( \alpha = am \) and get:

\[
\|\alpha - x\| = \|am - a/b\| = \|a/b\| \cdot \|bm - 1\| \leq \|bm - 1\| = \|np^i\| \leq 1/p^i
\]

Note that we can add multiples of \( p^i \) to \( \alpha \) and still have

\[
\|\alpha - k \cdot p^i - x\| \leq \max(1/p^i, 1/p^j) \leq 1/p^i.
\]

Therefore we can assume that \( \alpha \in \{0, \ldots, p^i - 1\} \). \( \square \)

Proof of Theorem 4.2.1. At first we prove the uniqueness: If \( \{a_i'\} \) is a different sequence satisfying (1) and (2) and if there exists \( i_0 \) such that \( a_{i_0} \neq a_{i_0}' \), then \( a_i \neq a_i' (mod(p^{i_0})) \) for every \( i > i_0 \). Therefore \( \|a_i - a_i'\| > 1/p^i \) so \( \{a_i'\}, \{a_i\} \) are not equivalent.

Now we prove existence: Suppose we have a Cauchy sequence \( \{b_i\} \in \mathbb{Q}_p \), we want to find an equivalent sequence \( \{a_i\} \) with the above property. Let \( N_j \) be the number such that for every \( i, i' > N_j \) we have \( \|b_i - b_{i'}\| < p^{-j} \), and we can choose \( N_j \) to be strictly increasing with \( j \), and \( N_j > j \). Observe that \( \|b_i\| \leq 1 \) if \( i > N_1 \). Indeed, for all \( i' > N_1 \) we have that \( \|b_i - b_{i'}\| < 1/p \), \( \|b_i\| \leq \max(\|b_{i'}\|, \|b_i - b_{i'}\|) \) and for \( i' \to \infty \) we have that \( \|b_{i'}\| \to \|a_i\|_p \leq 1 \).

Now we use the lemma and get a sequence \( \{a_j\} \) when \( 0 \leq a_j < p^j \) such that \( \|a_j - b_N\| < p^{-j} \). We claim that \( \{a_j\} \) is equivalent to \( \{b_i\} \), and satisfies the conditions of the theorem. It is indeed satisfies the conditions as:

\[
\|a_{j+1} - a_j\| = \|a_{j+1} - b_{N_{j+1}} + b_{N_{j+1}} - b_{N_j} - (a_j - b_{N_j})\|
\]

\[
\leq \max(\|a_{j+1} - b_{N_{j+1}}\|, \|b_{N_{j+1}} - b_{N_j}\|, \|a_j - b_{N_j}\|) \leq p^{-j}
\]

So \( a_{j+1} - a_j \) has at least \( p^j \) as a common divisor as required.

Furthermore, for any \( j \) and any \( i > N_j \) we have

\[
\|a_i - b_i\| = \|a_i - a_j + a_j - b_{N_j} - (b_i - b_{N_j})\| \leq \max(\|a_i - a_j\|, \|a_j - b_{N_j}\|, \|b_i - b_{N_j}\|) \leq p^{-j}
\]

So \( \{a_i\} \sim \{b_i\} \). \( \square \)

Now, if we have some \( \{a\} \in \mathbb{Q}_p \) with \( \|a\| \geq 1 \) then there exists some \( m \) such that \( \|a \cdot p^m\| \leq 1 \) and we have numbers with negative powers. Therefore we can present
the \( p \)-adic numbers as:

\[
\mathbb{Q}_p := \left\{ \sum_{i=-k}^{\infty} a_i \cdot p^i \text{, where } a_i \in \{0 \ldots p^i - 1\} \right\}.
\]

We define the \textit{ring of integers} \( \mathbb{Z}_p \), denoted \( \mathbb{Z}_p := \{ x \in \mathbb{Q}_p | \|x\|_p \leq 1 \} \) or equivalently \( \mathbb{Z}_p := \{ \sum_{i=0}^{\infty} a_i \cdot p^i \text{, where } a_i \in \{0 \ldots p^i - 1\} \} \) or equivalently \( \mathbb{Z}_p := \mathbb{Z} \| \|_p \) - the closure of \( \mathbb{Z} \) with respect to the \( p \)-adic norm. Notice that \( \mathbb{Z}_p \) is indeed a ring and that the only invertible elements are \( x \in \mathbb{Z}_p \) with \( \|x\|_p = 1 \).

4.3. \( p \)-adic expansions. We want to write the \( p \)-adic expansions of elements \( q \) in \( \mathbb{Q} \). If \( q \in \mathbb{N} \), that's just writing its \( p \)-base expansion. For example, \( (126)_{10} = \cdots 002001 \).

Let \( x := \frac{m}{n} \) be some rational number, with \( (n, m) = 1 \). It is enough to describe the expansion when \( p \nmid n \) (that is, when \( x \in \mathbb{Z}_p \cap \mathbb{Q} \)) as otherwise we can multiply \( x \) by \( p^k \) for some \( k \), calculate the expansion, and move the point \( k \) places to the left.

We can’t take remainder of \( x \) modulo \( p \), as with integers. Instead, we can calculate the fraction \( x = \frac{m}{n} \in \mathbb{F}_{p^k} \) for \( k \in \mathbb{N} \). Thus, the expansion of \( x \) in \( \mathbb{Q}_p \) is calculated inductively:

- Write the digit \( x_0 := \left[ \frac{m}{n} \right] \in \mathbb{F}_p \).
- The nominator of the difference \( \frac{m}{n} - x_0 = \frac{m - n \cdot x_0}{n} \) is divisible by \( p \). Redefine our fraction to be \( x := \frac{1}{p} \cdot \left( \frac{m}{n} - x_0 \right) \), and continue inductively.

\textbf{Example 4.3.1.} Calculate \( \frac{1}{2} \in \mathbb{Q}_7 \). We start by solving the equation \( 2x_0 = 1(\text{mod}7) \). The answer is \( x_0 = 4 \). In the second step we calculate \( \frac{1}{2}(\frac{1}{2} - 4) = x_1 \). So \( 2 \cdot (7x_1 + 4) = 1(\text{mod}49) \). Therefore \( x_1 = 3 \). We continue by induction and get the required expansion.

Every ball in \( \mathbb{Q}_p \) is a disjoint union of \( p \) balls. For \( p = 2 \), the ball \( \mathbb{Z}_2 = B_c(0,1) = B_c(0,2) \) consists of numbers with no digits to the right of the point. It’s a disjoint union of two balls, \( B_0 \) and \( B_1 \) - where each \( B_i \) consists of all numbers ending with the digit \( i \). Similarly, \( B_0 = B_{00} \cup B_{01}, B_1 = B_{10} \cup B_{11} \), where the elements in \( B_{ij} \) end with the digits \( ij \). And so on.

This recursive structure implies \( p \)-adic integers are homeomorphic to the Cantor set.

\textbf{Exercise 4.3.2.} Show that \( \sum_{n=0}^{\infty} a_n \) converges in \( \mathbb{Q}_p \) \( \iff \) \( |a_n|_p \to 0 \).

\textbf{Exercise 4.3.3.} Show \( \mathbb{Z}_p \) is homeomorphic to the Cantor set as topological spaces, where the Cantor set has its usual topology induced from the real numbers. In particular this shows \( \mathbb{Z}_p \) is a compact set.
4.4. Inverse limits.

Definition 4.4.1. Let \( A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \ldots \) be a sequence of Abelian groups \( \{A_i\} \) together with a system of homomorphisms \( \{f_{ij} : A_j \to A_i \mid j > i\} \), such that \( f_{ik} = f_{ij} \circ f_{jk}, \forall i \leq j \leq k \). An inverse limit of a sequence of Abelian groups is defined by the collection of compatible sequences:

\[
\text{lim}_{\leftarrow} A_i = \{ a \in \prod_{i \in \mathbb{N}} A_i : f_{ij}(a_j) = a_i, \forall i \leq j \in \mathbb{N} \}.
\]

Exercise 4.4.2. Prove the following:

1. Let \( A_i := \mathbb{Z}/p^i\mathbb{Z} \), and let \( f_{ij} \) be the projection \( \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z} \). Show that \( \text{lim}_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}/p \) as a topological ring.
2. \( \mathbb{Q}_p \) is the localization of \( \mathbb{Z}_p \) by \( p \).
3. Prove that \( \mathbb{Q}_p \) is homeomorphic to the Cantor set minus a point.
4. Prove that \( \mathbb{Q}_p^n \) and \( \mathbb{Q}_p \) are homeomorphic.
5. Let \( U \subset \mathbb{Q}_p^n \) be an open set. Show that either \( U \) is homeomorphic to the Cantor set, or to Cantor set minus a point.

4.5. Haar measure and local fields. Let \( X \) be a topological space and let \( C_c(X) \) be the space of continuous functions of compact support on \( X \). Recall that the space of continuous linear functionals \( C_c(X)^* \) can be identified with the space of regular Borel measures on \( X \).

Theorem 4.5.1 (Haar’s theorem). Let \( G \) be a locally compact topological group. Then:

1. There exists a measure \( \mu \) on \( G \) with values in \( \mathbb{C} \) such that \( \mu(U) = \mu(gU) \) for any measurable set. Equivalently, there exists \( \phi \in C_c(G)^* \) such that for any \( g \in G \) we have that \( \phi(f) = \phi(f_g) \) where \( f_g(x) = f(g^{-1} \cdot x) \).
2. This measure is unique up to a scalar.

Exercise 4.5.2.

1. Prove Haar’s theorem for \( (\mathbb{Q}_p, +) \).
2. Given a Haar measure \( \mu \), we can define another invariant measure \( \mu_a(B) = \mu(aB) \) for any \( a \in \mathbb{Q}_p \). Show that \( \mu_a = |a| \cdot \mu \).

Definition 4.5.3. A local field is a non-discrete topological field which is locally compact.

Theorem 4.5.4. Any local field \( F \) is isomorphic as a topological field to one of the following:

1. \( \mathbb{R} \) or \( \mathbb{C} \) (if \( F \) is Archimedean).
2. A finite extension of \( \mathbb{Q}_p \) for some prime \( p \) (if \( F \) is non-Archimedean of characteristic 0).
(3) The field of formal Laurent series $\mathbb{F}_p((t)) = \{ \sum_{i=-k}^{\infty} a_i t^i \mid a_i \in \mathbb{F}_p \}$ for some prime $p$ and natural number $s$ (if $F$ is non-Archimedean of characteristic $p$).

Proof. The main steps of the proof are as follows:

(1) Using Haar’s theorem define a measure on $(F, +)$. Using this measure we define an absolute value, up to scalar multiplication, that is, we set $|a| = \alpha(a)$ where $\mu_a = \alpha(a) \mu$.

(2) We show that for every local field the absolute value which was defined in (1) defines its topology.

(3) We prove that every compact metric space is complete.

(4) Every local field of characteristic 0 contains $\mathbb{Q}$ and one of its completions. This means that $F$ contains $\mathbb{R}$ if it is archimedean, and $\mathbb{Q}_p$ if it is non-archimedean.

(5) We show that if $F$ has characteristic 0, then $F$ must be a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$. Otherwise, if it is a non algebraic extension it must be non-locally-compact.

(6) We show that any finite extension of $\mathbb{Q}_p, \mathbb{R}$ or $\mathbb{F}_q((t))$ is indeed a local field.

(7) For $\text{char}(F) \neq 0$ we show that $F$ contains a transcendental element, name it $t$, and show that it contains $\mathbb{F}_q((t))$. We show that $F$ must be a finite extension of $\mathbb{F}_q((t))$.

\[\Box\]

4.6. Some basic properties of $\ell$-spaces.

Definition 4.6.1. An $\ell$-space $X$ is a Hausdorff, locally compact and totally disconnected topological space.

Remark 4.6.2. We usually add the demand $X$ is $\sigma$-countable, that is it is the union of countably many compact spaces. Such a space is also sometimes called countable at $\infty$.

Exercise 4.6.3. Find a compact $\ell$-space $X$ and $U \subseteq X$ such that $U$ is not countable at $\infty$.

Exercise 4.6.4. Show the following:

(1) Any non-archimedean local field is an $\ell$-space.

(2) Finite products, and open or closed subsets of an $\ell$-space are $\ell$-spaces. Note that any subset of a totally disconnected topological space is totally disconnected.

Proposition 4.6.5. Let $X$ be an $\ell$-space, then it has a basis of clopen (that is closed and open) sets (i.e. it is zero-dimensional).
Proof. Taken from [AT08, 3.1.7]. Assume we have a point \( x \in W \subseteq K \), with \( W \) open and \( K = \overline{W} \) compact and set \( \mathcal{P}_x = \{ U \subseteq K : U \text{ clopen in } K \text{ and } x \in U \} \) and \( P = \bigcap_{V \in \mathcal{P}_x} V \). Note that \( K \in \mathcal{P}_x \), thus \( P \neq \emptyset \).

Now, we claim that for every closed subset \( F \) of \( K \) such that \( F \cap P = \emptyset \) there exists some \( W \in \mathcal{P}_x \) such that \( W \cap F = \emptyset \). Indeed, set \( \eta = \{ U \cap F : U \in \mathcal{P}_x \} \). By assumption, it is a family of non-empty closed subsets of \( F \), and since \( F \) is compact if \( \bigcap_{V \in \eta} V = \emptyset \), then there is a finite collection of \( V_i \) such that \( \bigcap_{i=0}^{n} V_i = \bigcap_{i=0}^{n} U_i \cap F = \emptyset \) (note that this is an equivalent characterization of compactness via closed sets).

Now set \( W := \bigcap_{i=0}^{n} U_i \in \mathcal{P}_x \). Since \( \mathcal{P}_x \) is closed under finite intersections, \( W \in \mathcal{P}_x \).

Now we wish to show that \( P = \{ x \} \). Assume the contrary, i.e. \( P \neq \{ x \} \). \( P \) is disconnected since \( X \) is totally disconnected, so there exists non-empty closed \( x \in A \) and \( B \) such that \( A \cup B = P \) and \( A \cap B = \emptyset \) which are open in \( K \). Since \( K \) is regular, (Hausdorff + locally compact implies regular), there exist open disjoint sets \( A \subseteq U \) and \( B \subseteq V \) in \( K \), where we have \( F = K \setminus (U \cup V) \) closed in \( K \) and \( P \cap F = \emptyset \). We showed that for such \( F \) we can find \( W \in \mathcal{P}_x \) such that \( F \cap W = \emptyset \).

Now, observe that the open set \( G = U \cap W \) is also closed in \( K \) as,

\[
\overline{G} = \overline{U} \cap W \subseteq (K \setminus V) \cap (K \setminus F) = K \setminus (V \cup F) \subseteq U.
\]

Therefore \( \overline{G} \subseteq U \cap W = G \) (\( W \) was closed). Since \( x \in G \), we have \( G \in \mathcal{P}_x \), but as \( G \cap B = \emptyset \), we get that \( P = A \cup B \) is not contained in \( G \), which is a contradiction, implying \( P = \{ x \} \).

Since for every open set \( x \in O \) in \( K \) the set \( K \setminus O \) is compact and \( x \notin K \setminus O \), it follows from the above claim that \( O \) contains some \( V \in \mathcal{P}_x \).

Now, given an open set \( x \in A \) in \( X \), we have that \( W \cap A \subseteq K \cap A \), is open in \( K \), and thus contains a clopen \( U \) in the topology of \( K \cap A \) from the above. Now, \( U \) is closed in \( K \) and thus closed in \( X \), and open in \( \overline{W} \cap A \) but contained in \( W \cap A \) and thus open in \( X \). This finishes the proof.

take a compact neighborhood \( x \in B \) and use the regularity of \( X \) to obtain an open neighborhood \( V \) such that \( \overline{V} \subseteq A \cap B \).

\[ \square \]

**Exercise 4.6.6.** Show that every \( \sigma \)-compact, first countable \( \ell \)-space \( X \) is homeomorphic to one of the following:

1. Countable (or finite) discrete space.
2. Cantor set.
3. Cantor set minus a point.
4. Disjoint union of (2) or (3) with (1).

**Definition 4.6.7.** A refinement of an open cover \( \bigcup_{i \in I} U_i = X \) is an open cover \( \{ V_j \}_{j \in J} \) such that for any \( j \), we have that \( V_j \subseteq U_i \) for some \( i \).
Exercise 4.6.8.

(1) Let $C \subseteq X$ be a compact subset of an $\ell$-space. Then any open cover has an open compact disjoint refinement.

(2) Let $X$ be a $\sigma$-countable $\ell$-space, then any open cover has an open compact disjoint refinement.

Distributions on $\ell$-spaces.

Definition 4.6.9. Let $X$ be an $\ell$-space. A function $f : X \to \mathbb{C}$ is said to be smooth if it is locally constant, that is for every point $x \in X$ there is an open neighborhood $x \in U \subseteq X$ such that the restriction $f|_U$ is constant. Similarly to the archimedean case, the space of smooth functions on $X$ is denoted by $C^\infty(X)$.

Proposition 4.6.10. Let $X$ be an $\ell$-space. Show that smooth functions separate the points in $X$. Assuming this, the Stone-Weierstrass theorem implies that $C^\infty(X)$ is dense in the space of all continuous functions $C(X)$.

Proof. Let $x, y \in X$. As $X$ is Hausdorff and has a basis of open compact, sets there exists disjoint $U_x$ and $U_y$ which are compact and open. Set $f|_{U_x} = 1$ and $f|_{X \setminus U_x} = 0$. Then $f$ is smooth and $f(x) = 1$, and $f(y) = 0$. □

Definition 4.6.11. The space of smooth functions with compact support, $C^\infty_c(X) \subseteq C^\infty(X)$, are called Schwartz functions. We denote them by $S(X)$. We also denote $\text{Dist}(X) = C^\infty_c(X)^* = S^*(X)$. We consider both spaces as vector spaces without any topology.

Exercise 4.6.12. Let $X$ be an $\ell$-space, show that $S^*(X)$ is a sheaf.

Remark 4.6.13. In $\mathbb{R}^n$, the Schwartz functions are the functions whose derivatives decrease faster than any polynomial, and there is a strict containment $C^\infty_c(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$. We will define them in the next lectures.

4.7. Distributions supported on a subspace. Recall that over $\mathbb{R}$, the description of distributions on a space $X$ that are supported on a closed subspace $Z$ is complicated (we did that using filtrations). Distributions on $\ell$-spaces behave much better.

Definition 4.7.1. Let $X$ be an $\ell$-space, we define the support of a distribution $\xi \in S^*(X)$ as we did for distributions on real spaces, by $\text{supp}(\xi) = \bigcap_{\beta \in Z} D_\beta$, where $D_\beta \subset X$ are taken to be closed.

Proposition 4.7.2. (Exact sequence of an open subset). Let $U \subseteq X$ be open and set $Z = X \setminus U$. Then $0 \to S(U) \to S(X) \to S(Z) \to 0$ is exact.

Proof. It is clear that extension by zero $S(U) \to S(X)$ is injective, we show that $S(X) \to S(Z)$ is onto. Let $f \in S(Z)$. As $f$ is locally constant and compactly
supported, we may assume that $Z$ is compact and has a covering by a finite number of open sets $U_i$ (open in $Z$) with $f|_{U_i} = c_i$. Notice that each $U_i$ is of the form $U_i = W_i \cap Z$, where $W_i$ is open in $X$. Therefore, $Z \subseteq \bigcup_{i=1}^n W_i$, and as $Z$ is compact, we may refine $\{W_i\}_{i=1}^n$ and get that $Z \subseteq \bigcup_{j=1}^m V_j$ where $V_j$ are open, compact mutually disjoint and $V_j \cap Z \subseteq W_i \cap Z = U_i$ for some $i$. We can thus extend $f$ by setting $f(x) = c_i$ for $x \in V_j \subseteq W_i$ and zero otherwise. It is left to prove exactness at $S(X)$. Let $f \in S(X)$ such that $f|_Z = 0$. As $f$ is locally constant, there is an open set $Z \subseteq V$ such that $f|_V = 0$. This implies that $f$ is supported on $Z^c = U$ and therefore $f|_U \in S(U)$. 

**Corollary 4.7.3.** Let $X$ be an $l$-space, and $Z \subset X$ a closed subspace. Then:

1. The inclusion $i : S^*(Z) \to S^*_Z(X)$ is an isomorphism.
2. There is an exact sequence $0 \to S^*(Z) \to S^*(X) \to S^*(X\backslash Z) \to 0$.

**Remark 4.7.4.**

1. Note that if we replace $X$ by $\mathbb{R}$, then the map $i$ is not onto. For example, for $Z := \{0\} \subset \mathbb{R}$, the derivatives $d^2_0(x) \in S^*_Z(\mathbb{R}^2)$ but they are not in the image of $i$ as in that case $S^*(Z) \simeq \mathbb{C}$.
2. This can be corrected by replacing $S^*(X)$ by $S^*_Z(X)$. Thus the following is an exact sequence:

$$0 \to S^*_Z(X) \to S^*(X) \to S^*(X\backslash Z).$$

**Exercise 4.7.5.** Let $V$ be a vector space (not necessarily finite dimensional) over a field $K$, and $L \subset V$ a linear subspace. Show that $\forall f \in L^* \exists g \in V^*$ such that $g|_L \equiv f$. Use Zorn’s lemma.

**Proposition 4.7.6.** Let $X,Y$ be $\ell$-spaces. Given $f_1 \in S(X)$ and $f_2 \in S(Y)$, consider the bilinear map $\phi : S(X) \otimes S(Y) \to S(X \times Y)$ via $(\phi(f_1 \otimes f_2))(x,y) := f_1(x) \cdot f_2(y)$. Then $\phi$ is an isomorphism of vector spaces.

**Proof.** It is easy to see that the image lies in the space of locally constant functions. We first prove this map is surjective. Let $f \in S(X \times Y)$, then $f = \sum c_{U \times V} \in \operatorname{Im}\phi$ we are done. To show $\phi$ is injective, assume that $\phi(\sum f_{i1} \otimes f_{2i})(x,y) := \sum f_{i1}(x) \cdot f_{2i}(y) = 0$. We can assume that $\{f_{i1}\}$ are linearly independent and that $\{f_{2i}\}$ are non zero and that the sum is minimal with respect to these demands. If we take some $y$ such that $f_{2i}(y) \neq 0$ we get that for any $x \in X$, we have that $\sum f_{i1}(x) \cdot f_{2i}(y) = 0$. This implies that $f_{i1}(x)$ are linearly dependent. Contradiction. Hence $f_{2i} \equiv 0$ for all $i$, implying $f_{i1} \otimes f_{2i} \equiv 0$, contradicting the assumption that the sum is minimal. □
5. Lecture 5: Vector valued distributions

Definition 5.0.1. Let $F$ be either $\mathbb{R}$ or $\mathbb{C}$, let $X$ be a locally compact space and let $V$ be a vector space over $F$. We define $C^\infty_c(X,V)$ to be the space of smooth functions on $X$ with compact support with values in $V$. Here the smoothness of a function is the usual coordinate-wise one.

Exercise 5.0.2. Let $V$ be a topological vector space over $F$. Prove that $C^\infty_c(X,V)$ isomorphic to $C^\infty_c(X) \otimes_F V$ as topological vector spaces, where the topology on $C^\infty_c(X) \otimes_F V$ is given by choosing a basis to identify $V$ with $F^n$ and by then taking the product topology on $C^\infty_c(X) \otimes_F F^n \cong \text{can}(C^\infty_c(X))^n$. In particular, this topology is independent of a choice of a basis.

5.1. Smooth measures. Recall that a measure is a $\sigma$ additive map from the $\sigma$-algebra of Borel subsets of $X$ into $\mathbb{R}$. For us, the following characterization is better:

Definition 5.1.1. Let $X$ be a locally compact topological space. The space of signed measures on $X$ is $C^c_c(X)^*$, i.e. all continuous functionals on $C^\infty_c(X)$ (and all linear functionals if $X$ is an $\ell$-space). A signed measure is a measure if it is non-negative on non-negative functions.

As the space $C^c_c(X)$ is larger than $C^\infty_c(X)$, its dual is smaller. Explicitly, $C^c_c(X)^* \subseteq C^\infty_c(X)^*$ where the inclusion is the dual of the dense embedding $C^\infty_c(X) \hookrightarrow C^c_c(X)$. In $C^c_c(X)^*$ there is a one-dimensional space of Haar measures, which for $X = \mathbb{R}^n$ is just the space of multiples of the Lebesgue measure.

Remark 5.1.2. We usually consider the space of complex valued measures. As in the real case, it can naturally be identified with $C^c_c(X)^*$.

Definition 5.1.3. Let $V$ be a locally compact vector space (note that it must be finite dimensional as otherwise it is not locally compact). The space of Haar measures on $V$, denoted $\text{Haar}(V) \subseteq C^c_c(V)^*$, is the space of translation invariant measures (which exists by Haar’s theorem).

The fact that this space is one dimensional is non-trivial, but the intuition is as follows: A Borel measure on $V$ is determined by its value on cubes with rational coordinates, as they form basis of the topology. It is not hard to see that if the measure is translation invariant, the measures of these cubes are determined by the measure of the unit cube.

Exercise 5.1.4 (*). Let $V$ be a vector space over a local field and let $\xi \in C^\infty_c(V)^*$ be translation invariant. Prove that $\xi$ is a Haar measure, i.e. show it is a measure (note that $C^\infty_c(V)^* \supseteq C^c_c(V)^*$ so a-priori there might be translation invariant distributions which are not measures).
Definition 5.1.5. Let $V$ be a topological vector space. A measure $\mu$ on $V$ is called a smooth measure if $\mu \in C^\infty(V, \text{Haar}(V))$, i.e. $\mu = f(x)h$ where $f$ is smooth and $h$ is a Haar measure. We denote this space by $\mu^\infty(V)$, and the space of all compactly supported measures inside it by $\mu^\infty_c(V)$.

Remark 5.1.6. Note that by definition $\mu^\infty_c(V) \simeq C^\infty_c(V) \otimes \text{Haar}(V)$ canonically. We also have that $\mu^\infty_c(V) \simeq C^\infty_c(V)$ by choosing a Haar measure. This isomorphism is not canonical.

5.2. Generalized functions versus distributions. We are now in position to understand the difference between generalized functions and distributions. A distribution on $V$ is a continuous functional on the space of smooth functions with compact support:

$$\text{Dist}(V) := C^\infty_c(V)^*.$$ 

A generalized function is a continuous functional on the space of smooth measures with compact support on $V$, i.e.

$$C^{-\infty}(V) := C^\infty_c(V, \text{Haar}(V))^*.$$ 

As functions can be integrated against smooth measures of compact support, we have a bilinear pairing

$$C^\infty_c(V, \text{Haar}(V)) \times C^\infty_c(V) \to \mathbb{C}.$$ 

Thus we have the following picture:

$$
\begin{array}{c}
C^{-\infty}(V) \\ j \uparrow \hspace{1cm} i \uparrow \\
C^\infty_c(V) & \simeq & \mu^\infty_c(V) \\
\end{array}
$$

where the diagonals are dual to each other. Both inclusions $i$ and $j$ are obtained via the pairing $\langle \cdot, \cdot \rangle$.

Exercise 5.2.1. Show that $\text{Haar}(V) \simeq (\text{Dist}(V))^V$ or equivalently that $(\text{Dist}(V))^V$ is one dimensional, for any finite dimensional vector space $V$ over a local field $F$.

Definition 5.2.2. We can also define generalized functions with values in a vector space by either:

1) $C^{-\infty}(V, E) := C^{-\infty}(V) \otimes E$

2) $C^{-\infty}(V, E) := C^\infty_c(V, \text{Haar}(V) \otimes E^*)^*$

and then $C^{-\infty}(V, \text{Haar}(V)) := C^{-\infty}(V) \otimes \text{Haar}(V) = C^\infty_c(V)^* = \text{Dist}(V)$.

Exercise 5.2.3.

1) Show that the two definitions of $C^{-\infty}(V, E)$ are equivalent.
2) Describe an embedding $C^\infty_c(V, E) \hookrightarrow C^{-\infty}(V, E)$. 

5.3. Some linear algebra. Let $V$ be a finite dimensional vector space over a local field $F$, we define the exterior algebra as $\Lambda(V) = \bigoplus_{i=0}^{\dim V} \Lambda_i(V)$, where $\Lambda_i(V) = \bigotimes_{j=0}^{k} V/J$ and $J$ is the ideal generated in $\bigotimes_{j=0}^{k} V$ by the set $\{v_1 \otimes \ldots \otimes v_k : v_i = v_j \text{ for some } i \neq j\}$. Note that this implies that the elements of the exterior algebra are anti-symmetric (i.e. $v \otimes u = -u \otimes v$), and that $\Lambda^k(V) = 0$ if $k > \dim V$, since after choosing a basis for $V$ and decomposing an element in $\Lambda^k(V)$ to basic tensors, there must be a basis element which appears at least twice.

**Definition 5.3.1.** Let $V$ be an $n$-dimensional vector space over a local field $F$ with absolute value $|\cdot|$.

1. We define the space of $k$-forms $\Omega^k(V) = \Lambda^k(V^*)$.
2. For a 1-dimensional space $V$ we define a real vector space $|V| := \{f: V^* \to \mathbb{R} : \forall \alpha \in F, f(\alpha v) = |\alpha| f(v)\}$.
3. We define the densities of $V$ as $\text{Dens}(V) := \{f: V^n \to \mathbb{R} : f(Av) = |\det(A)| f(v)\}$.

Now, let $\Omega^{\text{top}}(V)$ be the space of anti-symmetric $n$-forms on $V$. It is a one-dimensional space, and $\Omega^{\text{top}}(V) = \Lambda^n(V^*)$.

**Exercise 5.3.2.** Let $B$ be the space of bases of $V$.

1. Show that $\Omega^{\text{top}}(V) = \{f: B \to F : f(B_1) = \det(M_{B_2}^{B_1}) f(B_2) \forall B_1, B_2 \in B\}$ where $M_{B_2}^{B_1}$ is the respective base changing matrix.
2. Show that $\Omega^{\text{top}}(V) = \{f: V^n \to F : f(Av_1, \ldots, Av_n) = \det(A) f(v_1, \ldots, v_n)\}$.

**Definition 5.3.3.** For a finite dimensional real vector space $V$ define the orientation line

$$\text{Ori}(V) := \{f: B \to \mathbb{R} : f(B_1) = \text{sign}(\det(M_{B_2}^{B_1})) \cdot f(B_2)\}.$$ 

**Exercise 5.3.4.** Using the tensor product of the natural maps $\Omega^{\text{top}}(V) \to \text{Dens}(V)$ and $\Omega^n(V) \to \text{Ori}(V)$ show that $\Omega^{\text{top}}(V) = \text{Dens}(V) \otimes \text{Ori}(V)$.

Note that the orientation line is a linear space and not just two points as one is used to think about orientations. However, we have two distinguished points in $\text{Ori}(V)$, the two functions with absolute value 1. These are the usual orientations we are used to thinking about.

**Proposition 5.3.5.** Show that there is a canonical isomorphism $\text{Dens}(V) \simeq \text{Haar}(V)$.

**Proof.** A Haar measure can be viewed both as a functional on compactly supported, continuous functions and as a function on a Borel algebra. The absolute value of the determinant $|\det| : V^n \to \mathbb{R}$ is an element of the one dimensional space
|Ω^n(V)| (recall that for finite dimensional spaces \( V \simeq^{can} V^{**} \)). We have a canonical isomorphism by choosing a basis \( \{e_i\}_{i=1}^n \) for \( V \), and bijecting between the element \( \varphi \in |Ω^n(V)| \) such that \( \varphi(e_1, \ldots, e_n) = 1 \) with the Haar measure normalized such that it has the value 1 on the parallelogram spanned by the vectors \( \{e_i\}_{i=1}^n \). This is independent of choice of basis since given a different basis both elements would be multiplied by the same factor of \( |\det(M)| \), where \( M \) is the change of basis matrix with respect to these two bases.

\[ \varphi \]

\[ \text{Exercise 5.3.6.} \] Show the following:

\begin{enumerate}
\item \(|L \otimes M| = |L| \otimes |M|\) for two one dimensional vector spaces \( L \) and \( M \).
\item \(|Ω^{op}(V)| \simeq \text{Dens}(V)\).
\item If \( W \subseteq V \) then \( \text{Haar}(W) \otimes \text{Haar}(V/W) \simeq^{can} \text{Haar}(V)\).
\item If \( W \subseteq V \) then \( Ω^{op}(V) \simeq Ω^{op}(W) \otimes Ω^{op}(V/W)\).
\item If \( F = \mathbb{R} \), then \( \text{Ori}(V) = \text{Ori}(W) \otimes \text{Ori}(V/W)\).
\end{enumerate}

5.4. Generalized functions supported on a subspace. Let \( W \subseteq V \) be linear spaces. We showed that over a non-archimedean local field \( F \) we have that \( \text{Dist}_W(V) = \text{Dist}(W) \), and for \( F = \mathbb{R} \) we described \( \text{Dist}_W(V) \) for the case where \( V = \mathbb{R}^n \) and \( W = \mathbb{R}^k \). The goal now is to describe distributions on \( V \) supported on \( W \) for any real linear spaces \( W \subseteq V \). Recall we have defined a (non-exhausting) filtration \( V_m(W) \) on \( C_c^∞(V) \) by

\[ V_m(W) = \{ f \in C_c^∞(V) | \forall i \in \mathbb{N}_0^{n-k} \text{ where } |i| \leq m \text{ it holds that } \frac{\partial^if}{\partial x^i} |W = 0 \} \]

where \( \dim(V) = n \) and \( \dim(W) = k \). We then defined \( F_m(W) \subseteq \text{Dist}_W(V) \) by

\[ F_m(W) = (C_c^∞(V)/V_m(W))^* := \{ \xi \in \text{Dist}(V) | \langle \xi, f \rangle = 0 \text{ for any } f \in F_m^0(V) \} \]

Note that we have that \( F_m(W) = V_m(W)^⊥ \) where \( V_2^⊥ := (V_1/V_2)^* \) for two vector spaces \( V_2 \subseteq V_1 \). We want to describe \( F_m(W)/F_{m-1}(W) \) in canonical terms, that is such that the isomorphism will respect diffeomorphisms of \( V \) which preserve \( W \).

\[ \text{Theorem 5.4.1.} \] We have an isomorphism of vector spaces which commutes with diffeomorphisms of \( V \) which preserve \( W \):

\[ F_m(W)/F_{m-1}(W) \cong^{can} C_c^∞(W, \text{Sym}^m(W^⊥))^* \simeq \text{Dist}(W) \otimes \text{Sym}^m(V/W) \]

The proof of the theorem is based on the following lemma:

\[ \text{Lemma 5.4.2.} \] \( F_m(W)/F_{m-1}(W) \cong (V_{m-1}(W)/V_m(W))^* \).

\[ \text{Proof.} \] For any \( \varphi \in F_m(W) \), the restriction \( \varphi_{|V_{m-1}(W)} \) vanishes on \( V_m(W) \), and we send it to the induced functional on \( V_{m-1}(W)/V_m(W) \) which we denote by \( \tilde{\varphi} \). This is an injective morphism, since if \( \tilde{\varphi} = 0 \) then \( \varphi_{|V_{m-1}(W)} = 0 \) so \( \varphi \in F_{m-1}(W) \). Surjectivity follows from the Hahn-Banach theorem in the following way: any \( \varphi \in (V_{m-1}(W)/V_m(W))^* \) can be extended to \( \tilde{\varphi} \in (C_c^∞(V)/V_m(W))^* = F_m(W) \). Therefore \( [\tilde{\varphi}] + F_{m-1}(W) \mapsto \varphi \). \[ \Box \]
Hence, in order to prove the theorem it will be sufficient to prove that
\[ V_{m-1}(W)/V_m(W) \cong C^\infty_c(W, \text{Sym}^i(W^\perp)). \]
For this we do the natural thing— we attach to \( f \in V_{m-1}(W)/V_m(W) \) its \( i \)-th derivatives. Explicitly, we define:
\[ \Phi(f)(w)(v_1, \ldots, v_i) = \partial_{v_1} \cdots \partial_{v_i} f(w). \]
It is well defined as \( f \) vanishes identically on \( W \), so this form vanishes on all the tangential derivatives. It is injective since if \( \Phi(f) = 0 \) then \( f \) vanishes with all of its derivatives up to degree \( i \), so it is in \( V_m(W) \).

**Exercise 5.4.3.**

1. Finish the proof of the lemma— show that \( \Phi \) is onto, hence an isomorphism.
2. Show that the isomorphism \( F_m(W)/F_{m-1}(W) \cong C^\infty_c(W, \text{Sym}^m(W^\perp))^* \) is invariant with respect to diffeomorphism of \((V, W)\).
3. Find \( \xi \in \text{Dist}(V \setminus W) \) such that there is \( \eta \in \text{Dist}(V) \) such that \( \eta|_{V \setminus W} = \xi \). That is, show that the natural map \( \text{Dist}(V) \to \text{Dist}(V \setminus W) \) is not onto.

To get a similar result for generalized functions, we twist by the one dimensional space of Haar measures:
\[ F_m(W)/F_{m-1}(W) \cong C^\infty_c(W, \text{Sym}^m(W^\perp))^* = C^{-\infty}(W, \text{Sym}^m(W^\perp) \otimes \text{Haar}(W)). \]
Take \( G_m(W) = F_m(W) \otimes \text{Haar}(W)^* \subseteq C^{-\infty}(V) \). We get by the compatibility of tensor product and quotient the following:
\[ G_m(W)/G_{m-1}(W) \cong C^{-\infty}(W, \text{Sym}^m(W^\perp) \otimes \text{Haar}(W)) \otimes \text{Haar}(V)^* \]
\[ \cong C^{-\infty}(W, \text{Sym}^m(W^\perp) \otimes \text{Haar}(W) \otimes \text{Haar}(W)^*). \]
The next exercise shows that \( \text{Haar}(W) \otimes \text{Haar}(V)^* \) can be presented in a simpler manner:

**Exercise 5.4.4.** Let \( W \subseteq V \).

1. Show that \( \text{Haar}(W) \otimes \text{Haar}(V/W) \cong_{\text{can}} \text{Haar}(V) \).
2. Show that \( \text{Haar}(V)^* \cong_{\text{can}} \text{Haar}(V)^* \).
3. Conclude that \( \text{Haar}(W) \otimes \text{Haar}(V)^* \cong_{\text{can}} \text{Haar}(W^\perp) \).

We arrive at the following corollary, yielding the desired description for generalized functions.

**Corollary 5.4.5.** By the above argument it follows that:
\[ G_m(W)/G_{m-1}(W) \cong C^{-\infty}(W, \text{Sym}^m(W^\perp) \otimes \text{Haar}(W^\perp)). \]
6. Lecture 6: Manifolds

After understanding generalized functions on vector spaces, we move to understand generalized functions on spaces which locally look like vector spaces. For this we define the notion of a manifold.

Definition 6.0.1.

1. Let \( X \) be a topological space. A cover \( \{ U_i \}_{i \in I} \) is called locally finite, if for any \( x \in X \) there is a neighborhood \( V \) such that \( V \cap U_i \neq \emptyset \) only for finitely many \( i \in I \).
2. A topological space \( X \) is called paracompact, if any open cover has a locally finite refinement.
3. A topological manifold is a topological space \( X \) which is locally homeomorphic to \( \mathbb{R}^n \) and is Hausdorff and paracompact.

Exercise 6.0.2.

1. Find a space \( X \) which is locally homeomorphic to \( \mathbb{R}^n \) at every point and is paracompact but is not Hausdorff.
2. Find a space which is Hausdorff, locally isomorphic to \( \mathbb{R}^n \) but is not paracompact.

We now give a definition of a smooth manifold which is different than the usual definition in differential topology and uses sheaves of functions.

Definition 6.0.3. A sheaf of (K-valued) functions \( \mathcal{F} \) on a topological space \( X \) is an assignment \( U \mapsto \mathcal{F}(U) \subseteq \{ f : U \to K | f \text{ is continuous} \} \) for every open \( U \subset X \) such that:

1. \( \mathcal{F}(U) \) is an algebra with unity.
2. The restriction map \( \text{res}_U^V : \mathcal{F}(U) \to \mathcal{F}(V) \) for \( V \subseteq U \) is the usual restriction \( f \mapsto f|_V \).
3. (Gluing) For every open cover \( U = \bigcup_{i \in I} U_i \), if there exists functions \( f_i \in \mathcal{F}(U_i) \) such that \( f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \) for any \( i, j \in I \), then there exists \( f \in \mathcal{F}(U) \) s.t. \( f|_{U_i} = f_i \) for any \( i \in I \).

A sheaf of functions on \( X \) will be denoted by a pair \( (X, \mathcal{F}) \).

Remark 6.0.4. Note that the second demand implies the identity axiom, that is if \( \text{res}_U^X(f) = f|_{U_i} = 0 \) for an open cover \( \{ U_i \}_{i \in I} \) of \( X \) then \( f = 0 \).

Example 6.0.5. Continuous or smooth functions on a space \( X \) form a sheaf of functions.

Definition 6.0.6. Let \( X \) and \( Y \) be topological manifolds.

1. A morphism \( \varphi : (X, \mathcal{F}) \to (Y, \mathcal{G}) \) between sheaves of functions \( (X, \mathcal{F}) \) and \( (Y, \mathcal{G}) \) is a map \( \varphi : X \to Y \) such that \( \forall g \in \mathcal{G}(U) \) we have that \( g \circ \varphi|_{\varphi^{-1}(U)} \in \mathcal{F}(\varphi^{-1}(U)) \). In other words, it is a map \( \varphi^\# : \mathcal{G} \to \varphi_\ast \mathcal{F} \).
(2) A smooth manifold is a space with a sheaf of functions \((X, C^\infty(X))\), where \(X\) is a topological manifold and for every point \(x \in X\) there is an open neighborhood \(U\) such that \((U, C^\infty(U)) \simeq (\mathbb{R}^n, C^\infty(\mathbb{R}^n))\) as sheaves of functions. That is there are differentiable maps \(\varphi : U \to \mathbb{R}^n\) and \(\psi : \mathbb{R}^n \to U\) such that \(\varphi \circ \psi\) and \(\psi \circ \varphi\) are the identity maps of the respective topological spaces and \((\psi \circ \varphi)^\# = \text{Id}_{C^\infty(U)}\) and \((\varphi \circ \psi)^\# = \text{Id}_{C^\infty(\mathbb{R}^n)}\) are the respective identity maps of sheaves.

**Remark 6.0.7.** The usual definition of manifolds adds an atlas to the structure of \(X\), that is an open cover \(X = \bigcup_{i \in I} U_i\) with diffeomorphisms \(\phi_i : U_i \to \mathbb{R}^n\). We also demand that \(\phi_i \circ \phi_j^{-1}\) is differentiable, so it seems like an additional demand with respect to the definition above. Alas, further rumination shows that a pair of isomorphisms \(\varphi_i : (U_i, C^\infty(U_i)) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))\) and \(\varphi_j : (U_j, C^\infty(U_j)) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))\) implies that the following composition is an isomorphism:

\[
(\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j})^\# : (\mathbb{R}^n, C^\infty(\mathbb{R}^n))|_{\varphi_i(U_i \cap U_j)} \to (\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j})^\#_* (\mathbb{R}^n, C^\infty(\mathbb{R}^n))|_{\varphi_j(U_i \cap U_j)}.
\]

In particular, by the next exercise we can deduce that \(\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j}\) is smooth and is a diffeomorphism. Therefore Definition 6.0.6(2) is equivalent to the usual definition of a smooth manifold.

**Exercise 6.0.8.**

1. Show that \(C^\infty(\mathbb{R}^n, \mathbb{R}^k) = \{f : \mathbb{R}^n \to \mathbb{R}^k : f^*(\mu) \in C^\infty(\mathbb{R}^n) \forall \mu \in C^\infty(\mathbb{R}^k)\}\).

2. Let \(M\) and \(N\) be smooth manifolds. Show that 6.0.6(1) is equivalent to the usual definition of a morphism of smooth manifolds. That is, that a map \(f : M \to N\) is a smooth map of manifolds \(\iff\) it is a morphism of ringed spaces (where the sheaf is a sheaf of smooth functions).

**Remark 6.0.9.** Note that by a theorem of Whitney every \(n\)-dimensional manifold can be embedded in \(\mathbb{R}^{2n+1}\). Thus we can always think about smooth manifolds sitting in \(\mathbb{R}^N\) for \(N\) large enough.

### 6.1. Tangent space of a manifold.

There are several equivalent ways to define the tangent space of a smooth manifold \(M\) at a point \(x \in M\). We first give a categorical definition and then construct several objects which satisfy this definition.

**Definition 6.1.1.** We denote by ptMan the category of smooth pointed manifolds, that is the objects are tuples of a smooth manifold \(M\) and a point \(x \in M\) and the morphisms are smooth maps of manifolds which preserve the distinguished points.

**Definition 6.1.2.** A functor is..

**Definition 6.1.3.** A tangent space of a smooth manifold \(M\) at a point \(x \in M\) is a functor \(\text{Tan} : \text{ptMan} \to \text{Vect}\) from pointed smooth manifolds to vector spaces which satisfies the following conditions:
Example 6.2.2.

Let $\text{Definition 6.1.5.}$

Exercise 6.1.4.

There are several structures that satisfy the above conditions:

(1) The space of all paths $T_x(M) := \{ \gamma : ((-1, 1), 0) \to (M, x) \}$ modulo the relation $\gamma_1 \sim \gamma_2$ if there exists a neighborhood $U$ of $x$ and an isomorphism $\phi : U \to \mathbb{R}^n$ s.t. $(\phi \circ \gamma_1)'(x) = (\phi \circ \gamma_2)'(x)$. It is easy to check that this definition does not depend on the choice of $(\phi, U)$.

(2) The space of derivations $T_x(M) = \{ d : C^\infty(M) \to \mathbb{R} : d \text{ is linear}, d(f \cdot g) = df \cdot g(x) + f(x) \cdot dg \}$. Define $m_x := \{ f \in C^\infty(M) : f(x) = 0 \}$, and take $T_x(M) := (m_x/m_x^2)^*$. The constructions of the tangent space given above are equivalent.

Definition 6.1.5. Let $\phi : M \to N$ be a smooth map. The differential of $\phi$ at $x \in M$ is the map $d_x\phi : T_x(M) \to T_{\phi(x)}(N)$ defined by $d_x(\phi)(\gamma) := \phi \circ \gamma$ for an equivalence class of paths $[\gamma] \in T_x(M)$.

Exercise 6.1.6.

(1) Show the differential is well defined, i.e. it does not depend on the representative $\gamma \in [\gamma]$.

(2) Show that given manifolds $M, N$, and $K$ and maps $\phi : M \to N$ and $\psi : N \to K$, the differentials satisfy $d_x(\psi \circ \phi) = d_{\phi(x)}(\psi) \circ d_x(\phi)$.

6.2. Types of maps between smooth manifolds.

Definition 6.2.1. Let $\phi : M \to N$ be a smooth map between smooth manifolds.

(1) $\phi$ is an immersion if $d_x\phi$ is injective.

(2) $\phi$ is a submersion if $d_x\phi$ is surjective.

(3) $\phi$ is a local isomorphism or étale if $d_x\phi$ is an isomorphism.

(4) $\phi$ is an embedding if it is an immersion and there is a homeomorphism $M \cong \phi(M)$.

(5) $\phi$ is a proper map if for every compact $K \subset N$, the preimage $\phi^{-1}(K)$ is compact. In particular, in that case all the fibers of $\phi$ are compact in $M$.

(6) $\phi$ is a covering map if for every $x \in N$ there exists a neighborhood $U \subset N$, such that $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is locally a diffeomorphism, and $\phi^{-1}(U) \cong U \times D$ for some discrete set $D$.

Example 6.2.2.
(1) Let \( \phi : [-1, 1] \to \mathbb{R}^2 \) be a smooth path that slows to a stop at \( \phi(0) = (0, 0) \), but spends no time at \( (0,0) \). That is, \( \phi'(0) = (0,0) \), but \( \phi(x) \neq 0 \) for all \( x \neq 0 \) in some neighborhood \([-\varepsilon, \varepsilon]\) of 0. Such a \( \phi \) is locally injective at 0, but since \( d_0\phi = 0 \) it is not an immersion at 0.

(2) An immersion is not necessarily one-to-one. As an example, consider a self-intersecting path \( \phi : \mathbb{R} \to \mathbb{R}^2 \) with constant speed.

(3) Let \( L \) and \( D \) be finite dimensional linear spaces. The differential of a linear map \( \phi : L \to V \) is \( \phi \) itself. Thus, a one-to-one \( \phi \) will be an immersion, an onto \( \phi \) will be a submersion, and an isomorphism of linear space will be an étale map.

Exercise 6.2.3. Let \( M \) and \( N \) be smooth manifolds.

(1) Find a map \( \phi : M \to N \) which is an injective immersion, but is not an embedding.

(2) Show that every proper map which is an injective immersion is a closed embedding.

(3) Show that a proper map which is étale is a covering map, and that a covering map with finite fibers is proper and étale.

Definition 6.2.4. Let \( X \) and \( Y \) be topological spaces. A fiber bundle is a map \( p : X \to Y \), such that for every \( x \in Y \) there exists a neighborhood \( U \subseteq Y \) such that \( p^{-1}(U) \cong U \times Z \) for \( U \subseteq Y \) for some topological vector space \( Z \).

Exercise 6.2.5. Show that a proper submersion is a fiber bundle.

6.3. Analytic manifolds and vector bundles. We would like to be able to talk about manifolds for a general local field. In order to do so, for a non-archimedean local field \( F \) we introduce the notion of an analytic \( F \)-manifold.

Definition 6.3.1. Let \( F \) be a non-archimedean local field. An analytic \( F \)-manifold is a space \( M \) which is locally isomorphic to \( \mathcal{O}_F^n \) together with a sheaf of functions

\[
\text{An}(U) = \{ f : U \to F : \forall x \in U, \exists r > 0 \text{ s.t. } f|_{B_r(x)}(y) = \sum_{k \in \mathbb{N}^n} a_k(x - y)^k \},
\]

where \( B_r(x) \) is the ball of radius \( r \) around \( x \), and \( \vec{k} \) is a multi-index, thus \( (x-y)^\vec{k} = \prod_{i=0}^{n} (x_i - y_i)^{k_i} \).

Remark 6.3.2. By Exercise 4.6.8 there is no need to use partition of unity for \( F \)-analytic manifolds.

Example 6.3.3. There exist singular analytic manifolds, and any singular affine algebraic variety is an example for such a manifold.

Definition 6.3.4. Let \( M \) be a smooth manifold or a \( p \)-adic analytic manifold. A real vector bundle over \( M \) is a tuple \( (E, p) \) where \( E \) is a topological space and \( p : E \to M \) is a continuous surjection such that:
(1) For every \( x \in M \) we have a structure of a finite dimensional real vector space on \( p^{-1}(x) = V_x \).

(2) For every \( x \in M \) there exists an open \( x \in U \) and a local trivialization \( \varphi_U : V_x \times U \to p^{-1}(U) \) where \( \varphi_U \) is a homeomorphism (or diffeomorphism if \( M \) is a real smooth manifold) and \( p \circ \varphi_U(v, x) = x \) for all \( v \in V_x \).

(3) The maps \( v \mapsto \varphi_U(v, x) \) are linear isomorphisms.

If \( E \cong V \times M \) we say \((E, p)\) is a trivial bundle over \( M \).

**Exercise 6.3.5.** It is known that the Mobius strip \( M \) is not homeomorphic to the (finite) cylinder \( I \times S^1 \). By extending each segment \( I \) of \( M \) to \( \mathbb{R} \), we can define a bundle \( E \) over the manifold \( S^1 \). This way, points of \( E \) are such that the fiber over \( \theta \in S^1 \) is a line in \( \mathbb{R}^3 \) which intersects the \( z \)-axis with angle \( 0.5 \cdot \theta \). Define the vector bundle above rigorously and show it is not diffeomorphic to the trivial bundle \( S^1 \times \mathbb{R} \).

**Example 6.3.6.** The tangent bundle of \( M = S^1 \) is \( TS^1 \cong S^1 \times \mathbb{R} \). The tangent space at any point is one-dimensional, and changes smoothly as we move along the circle. However, on \( M = S^2 \) the tangent bundle is not isomorphic to \( S^2 \times \mathbb{R}^2 \). This happens since every vector field on \( S^2 \) vanishes (Hairy ball theorem).

**Definition 6.3.7.** Let \((M, E)\) be a \( k \)-dimensional real vector bundle and \( \pi \) be its projection. Given trivializing neighborhoods \( U \) and \( V \), and trivializations \( \varphi_U : U \times \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(U) \) and \( \varphi_V : V \times \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(V) \), one can consider \( \varphi_V^{-1} \circ \varphi_U : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k \). We can then write \( \varphi_V^{-1} \circ \varphi_U(x,v) = (x, g_{U,V}(v)) \) where \( g_{U,V} \in \text{GL}(\mathbb{R}^k) \). The maps \( g_{U,V} \) are called transition functions.

Notice that the set of transition functions \( g_{U,V} \), satisfy the cocycle conditions

\[
g_{U,U}(x) = \text{Id} \quad \text{and} \quad g_{U,V}(x)g_{V,W}(x) = g_{U,W}(x).
\]

Conversely, given a fiber bundle \((E, X, \pi)\) of degree \( k \) with a transition map in \( \text{GL}(\mathbb{R}^k) \) abiding the cocycle condition which acts in the standard way on the fiber \( \mathbb{R}^k \), there is an associated a vector bundle. This is sometimes taken as the definition of a vector bundle.

**Proposition 6.3.8.** Given a manifold \( M \), vector bundles \( \{(E_i, p_i)\}_{i=1}^n \) each of which with fiber of constant dimension \( m_i \), over it, and a functor \( F : \text{Vect}^n \to \text{Vect} \), we can construct a vector bundle \((F(E_1, \ldots, E_n), q)\) over \( M \).

**Proof.** First, take a cover \( \{U_a\}_{\alpha \in I} \) of \( M \) which is a local trivialization of \( E \) (that is, \( p^{-1}(U_a) \cong V \times U_a \)). Define the total space \( F(E) \) over each \( U_a \) by \( F(V) \times U_a \), where the surjection \( q \) will be projecting to \( M \), and glue every two pieces \( q^{-1}(U_\alpha) \) and \( q^{-1}(U_\beta) \) by setting \((v, x) \sim (g_{\alpha,\beta}(v), x)\) for every \( x \in U_\alpha \cap U_\beta \) and \( v \in V \), where \( g_{\alpha,\beta} = F(\varphi^{-1}_{U_\alpha} \varphi_{U_\beta}) \). Finally, note that for any two elements of the cover \( g_{\alpha,\beta}^{-1} = g_{\beta,\alpha} \), and that in order for our construction to be well defined we need to
show the cocycle condition, namely that $g_{\beta,\gamma}g_{\alpha,\beta} = g_{\alpha,\gamma}$ when restricted to triple intersections. This holds since

$$g_{\beta,\gamma}g_{\alpha,\beta} = F(\varphi_{U,\gamma}^{-1}\varphi_{U,\beta})F(\varphi_{U,\alpha}^{-1}\varphi_{U,\beta}) = F(\varphi_{U,\alpha}) = g_{\alpha,\gamma}.$$  

Note that if we want $F(E)$ to have a smooth structure we need to demand that $F$ preserves smooth maps. □

**Example 6.3.9.** Let $E_1$ and $E_2$ be two vector bundles over $M$. The direct sum $E_1 \oplus E_2$ is defined by applying our construction above to the direct sum functor $\oplus : \text{Vect}^2 \to \text{Vect}$.

**Exercise 6.3.10.** Find two non-isomorphic bundles $E$ and $E'$, such that $E \oplus F \cong E' \oplus F$ for a bundle $F$ (Hint: use vector bundles over $S^2$).

**Definition 6.3.11.** Let $M$ be a manifold.

1. The tangent bundle of $M$ is the disjoint union of its tangent spaces $TM = \bigcup_{x \in M} \{x\} \times T_x M$.

2. Given a submanifold $N \subseteq M$, and an embedding $i : N \to M$, we define the normal bundle to $N$ in $M$ to be $N_M^N := i^*(TM)/TN$, where $i^*$ is the pullback of the bundle $TM$ to $N$. Similarly, the conormal bundle to $N$ in $M$ is $CN_N^M := (N_M^N)^*$.

**Example 6.3.12.** For the sphere $N = S^2 \subset \mathbb{R}^3 = M$, the normal bundle at a point is the normal line to it (i.e. the line passing at the point and at zero). It is diffeomorphic to the trivial bundle on $N$.

**Definition 6.3.13.** For vector bundles $E_1, E_2$ over a manifold (smooth or $F$-analytic), we define the following:

1. $E_1^*$.

2. $E_1 \oplus E_2$.

3. $E_1 \otimes E_2$.

4. For an embedding $\varphi : E_1 \hookrightarrow E_2$, we define $E_2/E_1$.

5. $\bigwedge^k(E_1)$, $\text{Sym}^k(E_1)$.

6. We define the density bundle of $E_1$ by $\text{Dens}(E_1)$.

**Definition 6.3.14.** Let $M$ be either a smooth manifold or an $F$-analytic manifold, we define its density bundle by $\text{Dens}(M) = |\Omega^\text{top}(TM)|$, that is the density bundle of its tangent bundle.

6.4. **Sections of a bundle.** A set theoretic section of a function $f : X \to Y$ is a function $g : Y \to X$ s.t. $g \circ f = \text{id}_X$.

**Example 6.4.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection $f(x, y) = x$. One example of a section is $g(x) := (x, \sin x)$. 

In many cases, sections of bundles give rise to important concepts:

- A section of the tangent bundle of a manifold is a vector field.
- A section of the $k$-th exterior power of the cotangent bundle of a manifold is a differential form of degree $k$.
- A section of the density bundle is a density on the manifold.
- A section of the orientation bundle is a choice of an orientation on the manifold.

**Exercise 6.4.2.**

1. Show that every manifold has a Riemannian metric, i.e., an inner product on tangent spaces
   $$\langle \cdot, \cdot \rangle_p : T_p M \times T_p M \to \mathbb{R}$$
   which varies smoothly.

2. Let $M$ be a smooth $n$-dimensional Riemannian manifold, that is a smooth real manifold with a Riemannian metric. Construct explicitly a density over $M$, that is a smooth section of the density bundle over $M$. The density should respect coordinate changes, and be the standard density when $M$ is a linear space with the standard inner product.

**Remark 6.4.3.** We do not always have non-zero top differential forms on a manifold $M$, and the Mobius strip is an example of a manifold with no non-zero top differential form. However, we can always find a non-zero density on $M$. Since with a density we can define a measure on the manifold, we can define integration over manifolds.

**6.5. Another description of vector bundles.**

**Definition 6.5.1.** Let $V$ be a finite dimensional vector space and $X$ a topological space.

1. We define the constant sheaf $\mathcal{V}_X$ to be the sheafification of the constant presheaf, which assigns to every open set in $X$ the vector space $V$.

2. We say that a sheaf $\mathcal{F}$ over $X$ is locally constant if for every $x \in X$ there exists an open $x \in U_x$ and a finite dimensional vector space $V_x$ such that $\mathcal{F}|_{U_x} \cong \mathcal{V}_{U_x}$.

**Exercise 6.5.2.** Let $V$ be a finite dimensional vector space and $X$ a topological space.

1. Show that $\mathcal{V}_X(U)$ consists of the locally constant functions from $U$ to $V$.

2. Show that if $X$ is a $\sigma$-compact $\ell$-space then every locally constant sheaf $\mathcal{F}$ such that $\mathcal{F}_x \cong \mathcal{F}_y$ for all $x, y \in X$ is isomorphic to the constant sheaf.

Up to now we have used the Grothendieck definition of a sheaf. In some situations the following definition is more useful.
**Definition 6.5.3.** A Leray sheaf on $X$ is a pair $(E, p)$ such that $p : E \to X$ is a local homeomorphism.

**Theorem 6.5.4.** The category of Leray sheaves is equivalent to the category of Grothendieck sheaves.

*Proof.* Given a Leray sheaf $(E, p)$ we define a Grothendieck sheaf by

$$F(U) := \{ s : U \to p^{-1}(U) : f \text{ is continuous and } p \circ s = \text{Id}_U \}$$

with the obvious restriction maps. This is a sheaf of functions.

For the other direction, given a Grothendieck sheaf $F$, we define $E = \bigsqcup_{x \in X} F_x$ with the natural projection map $p : E \to X$. The basis of the topology of $E$ is given by $U_{s,V} = \{(x, (s)_x) : x \in V \}$ where $V \subseteq X$ is open and $s \in F(V)$. □

**Exercise 6.5.5.** Complete the proof by showing that this is indeed an equivalence of categories.

**Exercise 6.5.6.**

(1) Show that covering spaces correspond to locally constant sheaves, and that a covering space is trivial when it corresponds to a constant sheaf.

(2) Give an example of a locally constant sheaf arising from a covering space which is not constant.

7. Lecture 7: Distributions on analytic manifolds and on smooth manifolds

**Definition 7.0.1.** Let $E$ be an $F$-analytic one dimensional vector bundle over an $F$-analytic manifold $X$. Define a real vector bundle $|E|$ as follows. As a set define $|E| := \{(x,v) : x \in X, v \in |E_x|\}$ and define a topology by giving $C$ the discrete topology, so locally $E|_U \simeq U \times F$ and $|E||_U \simeq U \times |F| \simeq U \times \mathbb{C}$. Hence, a base for the topology is $V_{i,U,\alpha} = \varphi_i(U \times \{\alpha\})$ where $\varphi_i : U \times \mathbb{C} \to |E||_U$ and $\alpha \in \mathbb{C}$.

**Remark 7.0.2.** Note that $\tilde{p} : |E| \to X$ is a local homeomorphism as $V_{i,U,\alpha} \simeq U$ as a topological space. Hence $\tilde{p}$ is a Leray sheaf. Its corresponding Grothendieck sheaf is $\mathcal{F}(U) := \{\text{continuous sections } U \to \tilde{p}^{-1}(U)\}$. This is a locally constant sheaf $\mathbb{C}^X$ over $X$.

**Definition 7.0.3.** We can now define the density bundle over an $F$-analytic manifold $X$ in two ways:

**Def 1 (Leray):** $\text{Dens}(X) := |\Omega^0_{\text{top}}(X)|$.

**Def 2 (Grothendieck):**

$\text{Dens}(X)(U) := \{ \mu \in \text{Measures}(U) | \forall \varphi \in \mathcal{O}_F^U \to U, \text{there exists } f \in C^\infty(\mathcal{O}_F^U) \text{ such that } \mu = \varphi_*(f \cdot \text{Haar}) \}$. 

Lemma 7.0.4. Let \( \varphi : F^n \to F^n \) be an analytic diffeomorphism, let \( f \in C_c(F^n) \) and let \( \mu \) be a choice of a Haar measure on \( F^n \). Then

\[
\langle \mu, f \rangle =: \int f \, dx = \int (f \circ \varphi) |\det(D_x\varphi)| \, dx.
\]

Exercise 7.0.5. Show that the above definitions are equivalent.

7.1. Smooth sections of a vector bundle. In this subsection we assume that \( F = \mathbb{R} \) and we are dealing with smooth manifolds.

Definition 7.1.1. We define smooth functions on a manifold \( M \) with compact support and values in a vector bundle \((E, \pi)\) by:

\[
C_c^\infty(M, E) := \{ f : M \to E : \pi \circ f = \text{Id}_M \text{ and } \exists K \text{ compact such that } f|_K \in (m, 0) \}.
\]

Recall that \( C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) = \lim_{m \to \infty} C_c^{\infty}(\mathbb{R}^n, \mathbb{R}^k) \) where \( K_m \) is an increasing sequence of compact sets such that \( \bigcup_{n=1}^{\infty} K_m = \mathbb{R}^n \). We define a topology on \( C_c^\infty(M, E) \) using the topology on \( C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \):

Case 1- The trivial case: \( M \simeq \mathbb{R}^n \) and \( E \simeq \mathbb{R}^n \times \mathbb{R}^k \) with the projection to the first component. Note that continuous sections from \( \mathbb{R}^n \) to \( \mathbb{R}^n \times \mathbb{R}^k \) are just functions in \( C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \). Hence we give \( C_c^\infty(M, E) \) the topology of \( C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \).

Exercise 7.1.2. Show that the above definition is well defined, i.e. it does not depend on the isomorphism \( M \simeq \mathbb{R}^n \) and \( E \simeq \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \). In other words, show that:

1. Given a diffeomorphism \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) it induces a homeomorphism \( \varphi^* : C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k) \) via precomposition.

2. Given a smooth map \( \psi \in C^\infty(\mathbb{R}^n, \text{GL}_k(\mathbb{R})) \) we have that \( \psi_* : C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k) \) by \( \psi_* f = \psi \circ f \) is a homeomorphism.

Case 2- General case: We can choose trivializing \( \{U_i\}_{i \in I} \) such that \( M = \bigcup_{i \in I} U_i \) where \( \varphi_i : U_i \simeq \mathbb{R}^n \) and \( \psi_i : E|_{U_i} \simeq \mathbb{R}^n \times \mathbb{R}^k \). We have a surjective map

\[
\varphi : \bigoplus_{i \in I} C_c^\infty(U_i, E|_{U_i}) \to C_c^\infty(M, E)
\]

by summation where surjectivity follows from partition of unity. We define the quotient topology on \( C_c^\infty(M, E) \) according to the map \( \varphi \), that is, a set \( U \subseteq C_c^\infty(M, E) \) is open if \( \varphi^{-1}(U) \) is open in \( \bigoplus_{i \in I} C_c^\infty(U_i, E|_{U_i}) \), where the latter is endowed with the direct sum topology.

Proposition 7.1.3. The topology on \( C_c^\infty(M, E) \) is well defined. That is, the definition does not depend on the choice of the cover \( \{U_i\}_{i \in I} \) of \( M \).

Proof. We need to show that given a different cover \( \{V_\beta\}_{\beta \in J} \) of \( M \) which locally trivializes \( M \) and \( E \), we get the same topology.
Consider the cover \( \{ W_{\alpha,\beta} \} \) for \( W_{\alpha,\beta} = U_{\alpha} \cap V_{\beta} \) which refines both covers. We need to show that for the addition map,

\[
\bigoplus_{\alpha \in I} \bigoplus_{\beta \in J} C^\infty_r (W_{\alpha,\beta}, E|_{W_{\alpha,\beta}}) \xrightarrow{\oplus} \bigoplus_{\alpha \in I} C^\infty_r (U_\alpha, E|_{U_\alpha})
\]
a set in the range is open if and only if its preimage is open, where \( W_{\alpha,\beta} \subseteq U_{\alpha} \simeq \mathbb{R}^n \) and \( E|_{W_{\alpha,\beta}} \simeq E|_{U_\alpha} \simeq \mathbb{R}^k \). In order to show the above, it is enough to handle each case \( \bigoplus_{\beta \in J} C^\infty_r (W_{\alpha,\beta}, \mathbb{R}^k) \xrightarrow{\oplus} C^\infty_r (U_\alpha, \mathbb{R}^k) \simeq C^\infty_r (\mathbb{R}^n, \mathbb{R}^k) \) separately, since in the direct sum topology a set is open if all the injections \( D_i \hookrightarrow \bigoplus D_i \) are continuous (we furthermore assume our open sets are convex).

Given a basic open set \( U_{(L_m, \epsilon_m, B_m)} \subseteq C^\infty_r (U_\alpha, \mathbb{R}^k) \) where \( L_m \) are mixed differentiations, \( \epsilon_m \in \mathbb{R}_{>0} \) and \( B_m \) are compact sets such that \( \bigcup_{m=1}^{\infty} B_m = \mathbb{R}^n \), it is of the form \( U_{(L_m, \epsilon_m, B_m)} = \sum_{m \in \mathbb{N}} V_{L_m, \epsilon_m, B_m} \), where

\[
V_{L_m, \epsilon_m, B_m} = \left\{ f \in C^\infty_r (\mathbb{R}^n, \mathbb{R}^k) : \text{supp}(f) \subseteq B_m, \sup_{x \in \mathbb{R}^n} ||L_m(f)|| < \epsilon_m \right\}.
\]

Now, take a finite sum \( \sum f_\beta \in +^{-1}(U_{(L_m, \epsilon_m, B_m)}) \) for \( \sum f_\beta = f = \sum_{i=0}^{l} f_{m_i} \) and \( f_{m_i} \in V_{L_m, \epsilon_m, B_m} \). Let \( f_\beta = pr_\beta(f) \) be the projection of \( f \) into \( C^\infty_r (W_{\alpha,\beta}, \mathbb{R}^k) \), and define \( N = \# \{ \beta : pr_\beta(f) \neq 0 \} \) and \( \epsilon_m' = \frac{\epsilon_m - \sup_{m,i} ||L_m(f_{m_i})||}{N} \) and set \( \epsilon_m' = \frac{\epsilon_m'}{N} \) if \( m \neq m_i \) for all \( 0 \leq i \leq l \). For \( B'_{m,\beta} \subseteq W_{\alpha,\beta} \), compact sets which exhaust \( W_{\alpha,\beta} \) and such that \( B'_{m,\beta} \subseteq B_m \), the sets \( U_{(L_m, \epsilon_m', B'_{m,\beta})} \) are basic open sets in each \( C^\infty_r (W_{\alpha,\beta}, \mathbb{R}^k) \), and their direct sum is open in the direct sum. Now, we claim that,

\[
f \in \bigoplus_{\beta : f_\beta \neq 0} f_\beta + U_{(L_m, \epsilon_m', B'_{m,\beta})} \subseteq +^{-1}(U_{(L_m, \epsilon_m, B_m)}).
\]

Given \( g = \sum_{\beta : f_\beta \neq 0} g_\beta \) where \( g_\beta \in U_{(L_m, \epsilon_m', B'_{m,\beta})} \), then \( g_\beta = \sum_{i=1}^{l} g_{\beta,i} \), where \( g_{\beta,i} \in V_{L_{m_i}, \epsilon_m', B'_{m_i,\beta}} \).

Thus if \( n_i = m_i \) for some \( i \), we have \( \sup_{x \in B_{m_i}} ||L_m(g_{\beta,m_i})|| < \epsilon_m' = \frac{\epsilon_m - \sup_{m,i} ||L_m(f_{m_i})||}{N} \)

implying that,

\[
\sup_{x \in B_{m_i}} \left\| \sum_{\beta : f_\beta \neq 0} L_m(f_{\beta,\beta} + g_{\beta,m_i}) \right\| \leq \sup_{x \in B_{m_i}} \left\| \sum_{\beta : f_\beta \neq 0} L_m(f_{\beta,\beta}) \right\| + \sum_{\beta : f_\beta \neq 0} \sup_{x \in B_{m_i}} \left\| L_m(g_{\beta,m_i}) \right\|
\]

\[< \sup_{x \in B_{m_i}} \left\| L_m(f_{m_i}) \right\| + \sum_{\beta : f_\beta \neq 0} \left( \epsilon_m - \sup_{N} \left\| L_m(f_{m_i}) \right\| \right) = \epsilon_m.\]
Otherwise, if \( n_j \neq m_i \) for all \( i \), set \( n' = n_j \), and using the requirement \( \sup_{x \in B_{n'}} \| L_{n'}(g_{\beta,n'}) \| < \epsilon_n' \) we note that:

\[
\sup_{x \in B_{n'}} \left\| \sum_{\beta:j \neq 0} L_{n'}(g_{\beta,n'}) \right\| \leq \sum_{\beta:j \neq 0} \sup_{x \in B_{n'}} \| L_{n'}(g_{\beta,n'}) \| < N \frac{\epsilon_n'}{N} = \epsilon_n'.
\]

This allows us to conclude that \( f + g = \sum_{m=1}^{l} \sum_{i=1}^{l} f_{m,i} + \sum_{j=0}^{l} g_{\beta:i,j} \) lie in \( \bigoplus_{\beta \in J} C_c^\infty(W_{\alpha,\beta}, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \) are continuous (a cookie for the person who finds a quick proof for this), so it is enough to show that the addition is continuous. For a less cumbersome approach, note that the embeddings \( \bigoplus_{\beta \in J} C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \) is continuous. Since the domain has the direct sum topology, it is enough to check this for a finite direct sum, which follows by the continuity of addition in a topological vector space.

To show the map is open, it is enough to consider \( \bigoplus_{\beta \in J} C_c^\infty(W_{\alpha,\beta}, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \to \bigoplus_{\beta \in J} C_c^\infty(\mathbb{R}^n) \), for every compact \( K \), and since the domain has the direct sum topology and the basic open sets are finite sums of open sets in each coordinate, it is enough to show it for a finite direct sum \( \bigoplus_{i=1}^{m} C_c^\infty(W_i, \mathbb{R}^k) \to \bigoplus_{j=1}^{k} C_c^\infty(\mathbb{R}^n) \) where \( K \subset \bigcup_{i=1}^{m} W_i \). Now, use partition of unity \( f_i \), with \( C_i = \text{supp}(f_i) \subset W_i \) where \( \sum_{i=1}^{m} f_i \mid_K = 1 \) to get an onto map via the composition,

\[
\bigoplus_{i=1}^{m} C_c^\infty(W_i, \mathbb{R}^k) \to \bigoplus_{i=1}^{m} C_c^\infty(W_i, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n).
\]

Since this is a continuous surjective map of Fréchet spaces, it must be open, implying that the addition is open since the embedding is continuous.

We now give a different description of the topology of \( C_c^\infty(\mathbb{R}^n) \). Firstly observe that \( f \in C_c^\infty(\mathbb{R}^n) \iff \) for any \( g \in C_c^\infty(\mathbb{R}^n) \) we have that \( gf \) is bounded. Now let \( D \in \text{Diff}(\mathbb{R}^n) \) be a differential operators on \( C_c^\infty(\mathbb{R}^n) \). Define a seminorm \( \| f \|_D \) by \( \sup_{x \in \mathbb{R}^n} |D(f)(x)| \). We get that \( C_c^\infty(\mathbb{R}^n) \) is an inverse limit of Banach spaces \( B_D \) where each \( B_D \) is the completion of \( C_c^\infty(\mathbb{R}^n) \) with respect to \( \| \cdot \|_D \). (Verify with Rami).

**Definition 7.1.4.** Let \( M \) be a manifold and \( D : C^\infty(M) \to C^\infty(M) \) be a map. We say that \( D \) is a differential operator on \( M \) if for any trivializing cover \( \bigcup_{i \in I} U_i = M \)
and $\varphi_i : U_i \xrightarrow{\sim} \mathbb{R}^n$ we have that $\varphi_i^{-1} \circ D \circ \varphi_i \in \text{Diff}(\mathbb{R}^n)$. We denote the space of all differential operators on $M$ by $\text{Diff}(M)$.

We would like to define differential operators from $C^\infty(M, E)$ to $C^\infty(M, E')$, which we denote by $\text{Diff}(C^\infty(M, E), C^\infty(M, E'))$. As before we divide the definition into cases:

**Case 1- the trivial case:** Assume that $E \simeq M \times \mathbb{R}^k$ and $E' \simeq M \times \mathbb{R}^{k'}$. Then $\text{Diff}(C^\infty(M, E), C^\infty(M, E')) \simeq \text{Diff}(C^\infty(M)^k, C^\infty(M)^{k'})$ and the latter space is isomorphic as a vector space to the space of $k \times k'$ matrices with values in $\text{Diff}(C^\infty(M))$.

**Exercise 7.1.5.** Show that the definition of the space of differential operators $\text{Diff}(C^\infty(M, E), C^\infty(M, E'))$ does not depend on the isomorphisms $E \simeq M \times \mathbb{R}^k$ and $E' \simeq M \times \mathbb{R}^{k'}$.

**Case 2- the general case:** Let $A \in \text{Hom}(C^\infty(M, E), C^\infty(M, E'))$. Then we say that $A \in \text{Diff}(C^\infty(M, E), C^\infty(M, E'))$ if:

1. For any $f_1, f_2 \in C^\infty(M, E)$ such that $f_1|_U = f_2|_U$, we have that $A|_U = A|_U$.
2. If $E'|_U$ is a trivialization then $A|_U \in \text{Diff}(U, E|_U, E'|_U)$.

**Definition 7.1.6** (Second definition to the topology on $C^\infty_c(M, E)$). For $D \in \text{Diff}(C^\infty(M, E), C^\infty(M, E))$ define $\|f\|_D = \sup_{x \in M} |D(f)(x)|$. Define the topology on $C^\infty_c(M, E)$ via

$$C^\infty_c(M, E) = \lim_{\longrightarrow D}(C^\infty_c(M, E))_{\|\cdot\|_D}.$$ 

**Exercise 7.1.7.** Given a manifold $M$ and a vector bundle $E$ over it show that the two definitions of the topology on $C^\infty_c(M, E)$ are equivalent (one defined via taking a cover of $M$ and trivialization of $E$ and the other through differential operators).

**8. Lecture 8: Distributions over geometric objects**

**Definition 8.0.1.** Let $M$ be a smooth or $F$-analytic manifold, and let $E$ be a smooth vector bundle over it (in the case of an analytic manifold it has the discrete topology).

1. The space of distributional $E$-sections is defined to be $\text{Dist}(M, E) := C^\infty_c(M, E)^*$.
2. The space of generalized $E$-sections is defined to be $C^{-\infty}(M, E) = \text{Dist}(M, E^* \otimes \text{Dens}(M))$.

Although we do not have a natural injection from $C^\infty_c(M, E)$ to $C^\infty_c(M, E)^*$, we have a natural injection $i : C^\infty_c(M, E) \hookrightarrow C^{-\infty}(M, E)$ as follows: let $\mu \in C^\infty_c(M, E^* \otimes \text{Dens}(M))$ and $f \in C^\infty_c(M, E)$. Note that $f \otimes \mu \in C^\infty_c(M, E^* \otimes E \otimes \text{Dens}(M))$, that is, $f \otimes \mu(m) = f(m) \otimes \mu(m)$. Note that we have a natural map $q : C^\infty_c(M, E^* \otimes E \otimes \text{Dens}(M)) \to C^\infty_c(M, \text{Dens}(M))$ by pairing $E$ with $E^*$ and a natural map $\int : C^\infty_c(M, \text{Dens}(M)) \to \mathbb{C}$ by integrating over $M$ according to the
measure defined by the section of the density bundle. Hence \( \langle i(f), \mu \rangle = \int_M q(f \otimes \mu) \). Therefore, the definition of generalized sections indeed generalizes smooth sections.

**Proposition 8.0.2.** Let \( X \) be either a smooth or an \( F \)-analytic manifold. Then \( C_c^\infty(X)^\ast = C^{-\infty}(X) \).

**Proof.** Recall that \( C^{-\infty}(X) = \mu_c^\infty(X)^\ast \). Given a topological vector space \( V \), for \( W \subseteq V^\ast \) the space \( W \) is dense with respect to the weak topology if and only if \( W^\perp = \{ v \in V : \langle w, v \rangle = 0 \ \forall w \in W \} = \{ 0 \} \). To see the relevant direction, if \( W^\perp = \{ 0 \} \), we will show that for every \( \xi \in V^\ast \), finite set \( S \subset V \) and \( \epsilon > 0 \) we can find \( w \in W \) such that \( \xi|_S = w|_S \). Given such \( \xi \in V^\ast \), \( S = \{ v_1, \ldots, v_n \} \) and \( \epsilon > 0 \), assume \( S \) is a linearly independent set, and consider \( \rho : V^\ast \to \mathbb{R}^n \) by \( \rho(\eta) = (\langle \eta, v_1 \rangle, \ldots, \langle \eta, v_n \rangle) \). The map \( \rho|_W \) is onto, since otherwise there exist \( c_i \in \mathbb{R} \) such that \( \sum_{i=1}^n c_i \langle w, v_i \rangle = 0 \) for all \( w \in W \) (it must lie in some hyperplane, and all hyperplanes are of this form), but this means that \( \langle w, \sum_{i=1}^n c_i v_i \rangle = 0 \) implying \( \sum_{i=1}^n c_i v_i \in W^\perp = \{ 0 \} \). The surjectivity of \( \rho|_W \) allows us to find the desired \( w \in W \).

Thus it is enough to show that given \( \eta \in \mu_c^\infty(X) \), if \( \langle f, \eta \rangle = 0 \) for all \( f \in C_c^\infty(X) \) then \( \eta = 0 \).

Assume \( X \) is a smooth manifold. Given a non-zero measure \( \eta \), there exists some \( \mathbb{R}^n \simeq U \subset X \) such that \( \eta|_U \neq 0 \), to see this either use the fact that distributions form a sheaf, or view it as a positive function on Borel sets. Now, since \( U \simeq \mathbb{R}^n \) we must have that \( \eta|_U = g \cdot \mu_{\text{Haar}} \) where \( g \in C^\infty(\mathbb{R}^n) \). Taking some cutoff function \( \psi \in C_c^\infty(\mathbb{R}^n) \) such that \( \psi|_{B(0,\epsilon)} = 1 \) and \( \psi \geq 0 \) implies the desired result as \( \langle g \psi, \eta \rangle = \langle \psi \cdot g, \mu_{\text{Haar}} \rangle = \langle g^2 \psi, \mu_{\text{Haar}} \rangle > 0 \) as this is an integral of a positive function.

For an \( F \)-analytic manifold we do the same procedure only this time \( \psi \) is the indicator function of the open unit ball in \( F^n \). \( \square \)

**Exercise 8.0.3.** Let \( M, N \) be either smooth or \( F \)-analytic manifolds and let \( X, Y \) be \( \ell \)-spaces.

(1) Show that \( C_c^\infty(M) \otimes C_c^\infty(N) = C_c^\infty(M) \times C_c^\infty(N) \) and \( C_c^\infty(X) \otimes C_c^\infty(Y) = C_c^\infty(X) \times C_c^\infty(Y) \).

(2) Find an example such that \( C_c^\infty(X)^\ast \otimes C_c^\infty(Y)^\ast \nsubseteq C_c^\infty(X \times Y)^\ast \). Find an example that this isomorphism fails for The smooth and \( F \)-analytic manifolds (Hint: consider \( X = Y = \mathbb{Z} \)).

(3) Let \( E_1 \) and \( E_2 \) be complex vector bundles over \( M_1 \) and \( M_2 \) respectively. Show that \( C_c^\infty(M_1 \times M_2, E_1 \boxtimes E_2) = C_c^\infty(M_1, E_1) \otimes C_c^\infty(M_2, E_2) \).

**Definition 8.0.4.** Let \( X \) be an \( \ell \)-space and \( \mathcal{F} \) a sheaf over \( X \). Define \( \mathcal{F}_c(X) \) to be the space of compactly supported global sections of \( \mathcal{F} \), that is all \( s \in \mathcal{F}(X) \) such that \( s|_K = 0 \) outside some compact \( K \). Define \( C_c^\infty(X, \mathcal{F}) := \mathcal{F}_c(X) \) and \( \text{Dist}(X, \mathcal{F}) = C_c^\infty(X, \mathcal{F})^\ast \).
Theorem 8.0.5. Let $i: Z \hookrightarrow X$ be $\ell$-spaces where $Z$ is closed. Then:

1. $\text{Dist}(X, \mathcal{F})|_Z \simeq \text{Dist}(Z, \mathcal{F}|_Z) = i^*(\mathcal{F})$.
2. We have the following short exact sequence:

$$0 \to \text{Dist}(Z, \mathcal{F}|_Z) \to \text{Dist}(X, \mathcal{F}) \to \text{Dist}(U, \mathcal{F}|_U) \to 0.$$

We now want to prove the following important theorem:

Theorem 8.0.6. Let $N \subseteq M$ be a closed submanifold of a real manifold $M$, and let $E$ a bundle over $M$. Then there is a canonical filtration $F_i \subseteq \text{Dist}(M, E)$ such that:

1. Every $\xi \in F_i$ is supported on $N$.
2. $F_i$ is locally exhaustive, i.e. $\bigcup_{i=1}^{\infty} F_i$ is locally exhaustive.
3. $F_i/F_{i-1} \simeq \text{Dist}(N, E|_N) \otimes \text{Sym}^i(\text{CN}^M_N)$.

In order to prove the theorem, we would like to define the notion of derivatives of smooth sections $f \in C_c^\infty(M, E)$. Alas, the value of the derivative depends on the chart defined on $M$, so it is not well defined. Fortunately, the notion of vanishing of derivatives of certain order is well defined as the following exercise shows:

Exercise 8.0.7. Let $f \in C^\infty(\mathbb{R}^n)$ with $f^{(i)}(0) = 0$ for every $|i| < k$, and $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism such that $\varphi(0) = 0$. Furthermore let $g \in C^\infty(\mathbb{R}^n)$ be a nowhere vanishing function, and set $\tilde{f}(x) = f \circ \varphi^{-1}(x)g(x)$.

1. Show that:

$$\left( \frac{\partial^k}{\partial v_1 \cdots \partial v_k} \tilde{f} \right)(0) = \left( \frac{\partial^k}{\partial ((d\varphi)v_1) \cdots \partial ((d\varphi)v_k)} f \right)(0)g(0).$$

2. Find a counter example for part (1) if $f^{(i)}(0) \neq 0$ for some $|i| < k$.

Remark 8.0.8. As a consequence of this exercise, given any $f \in C^\infty_c(M, E)$ whose first $k-1$ derivatives vanish we can define the $k$-th differential symbol of $f$ denoted $d_x^k f : T_x M \times \cdots \times T_x M \to E_x$ by

$$d_x^k f(\xi_{1,i}, \ldots, \xi_{k,i}) = \left( \frac{\partial^k}{\partial \xi_{1,i} \cdots \partial \xi_{k,i}} (f \circ \varphi_i^{-1}) \right)(0),$$

where $\varphi_i$ is a local chart and $\xi_{1,i} = (\varphi_i \circ \gamma_i)'(0)$ are tangent vectors. If we choose a different chart $\varphi_j$ we get that

$$d_x^k f(\xi_{1,j}, \ldots, \xi_{k,j}) = \left( \frac{\partial^k}{\partial \xi_{1,j} \cdots \partial \xi_{k,j}} (f \circ \varphi_j^{-1}) \right)(0) = \left( \frac{\partial^k}{\partial \xi_{1,j} \cdots \partial \xi_{k,j}} (f \circ \varphi_i^{-1} \circ \varphi) \right)(0)$$

where $\varphi := \varphi_i \circ \varphi_j^{-1}$. By the discussion above, we get that

$$\left( \frac{\partial^k}{\partial \xi_{1,j} \cdots \partial \xi_{k,j}} (f \circ \varphi_j^{-1}) \right)(0) = \left( \frac{\partial^k}{\partial (d\varphi)\xi_{1,j} \cdots \partial (d\varphi)\xi_{k,j}} (f \circ \varphi_i^{-1}) \right)(0),$$
Proof. We have that:

\[\varphi(x_1, \ldots, x_k) = d_x \varphi \cdot (\varphi_1 \circ \gamma_1)'(0) = (\varphi \circ \varphi_i \circ \gamma_i)'(0) = \xi_{1,i}\]

we have that \(d_x f(x_1, \ldots, x_k) = d_x f(\xi_1, \ldots, \xi_k)\) so this is well defined.

We can now prove Theorem 8.0.6:

**Proof.** Note that we can identify \(d^k_x f \in \text{Sym}^k(T_x^* M) \otimes E_x\). Let \(N \subseteq M\) be a submanifold. Define:

\[F_i^k(C^\infty(M, E)) = \{ f \in C_c^\infty(M, E) : \forall x \in N, \text{ the first } k - 1 \text{ derivatives of } f \text{ vanish}\}.

Choose trivializations \(M|_U \simeq \mathbb{R}^n\) and \(N|_{U \cap N} \simeq \mathbb{R}^k\), then we showed that \(F_{V}^i(V)/F_{W}^i(V) \simeq C_c^\infty(W, \text{Sym}^i(W^\perp))\) using the map \(f \mapsto D^k_x f\). Hence we get that:

\[F_i^k/N_i^k \simeq C_c^\infty(N, E|_N \otimes C \text{ Sym}^i(C^N_N)).\]

This gives a canonical filtration \(F_i \subseteq \text{Dist}_N(M, E)\) such that

\[F_i/F_{i-1} \simeq (F_i^k/N_i^k)^* \simeq C_c^\infty(N, E|_N \otimes C \text{Sym}^i(C^N_N))^* = \text{Dist}(N, E|_N \otimes C \text{Sym}^i(C^N_N)).\]

\[\square\]

**Corollary 8.0.9.** We have that

\[\text{Gr}_i(C^{-\infty}(M, E)_N) = C^{-\infty}(N, E|_N \otimes \text{Dens}(M)^*|_N \otimes \text{Sym}^i(N_M^N)) \otimes \text{Dens}(N)).\]

**Proof.** We have that:

\[\text{Gr}_i(C^{-\infty}(M, E)_N) = \text{Gr}_i(\text{Dist}_N(M, E^* \otimes \text{Dens}(M)))\]

\[\simeq \text{Dist}(N, E^*|_N \otimes \text{Dens}(M)|_N \otimes \text{Sym}^i(C^N_N)) = C^{-\infty}(N, E|_N \otimes \text{Dens}(M)^*|_N \otimes \text{Sym}^i(N_M^N)) \otimes \text{Dens}(N)).\]

\[\square\]

### 8.1. Operations on generalized functions.

At this subsection we assume \(X\) and \(Y\) are either \(\ell\)-spaces, analytic \(F\)-manifolds (with or without complex bundles over them), or smooth manifolds.

**Definition 8.1.1.** Let \(\varphi : X \to Y\) be a map. We define the pullback \(\varphi^* : C^\infty(Y) \to C^\infty(X)\) by \(\varphi^*(f) = f \circ \varphi\). It is easy to see that if \(\varphi\) is proper then \(\varphi^* : C_c^\infty(Y) \to C_c^\infty(X)\). By dualizing, we get an operation \(\varphi_* : \text{Dist}(X) \to \text{Dist}(Y)\) on distributions to which we call pushforward by \(\varphi_* (\xi)(f) := \xi(\varphi^*(f)) = \xi(f \circ \varphi)\).

Note that if \(\varphi\) is not proper then can define \(\varphi_* : \text{Dist}(X)_\text{prop} \to \text{Dist}(Y)\) where \(\text{Dist}(X)_\text{prop} := \{ \xi \in \text{Dist}(X) | \varphi|_{\text{supp}(\xi)} \text{ is proper}\}\). We would like to set \(\langle \varphi_* \xi, f \rangle = \langle \xi, f \circ \varphi \rangle\), but \(f \circ \varphi\) might not be compactly supported. Therefore we choose a cutoff function \(\rho\) such that \(\rho|_{\text{supp}(\xi)} = 1\) and \(\rho|_{U \cap U} = 0\) where \(U\) is a small neighborhood of \(\text{supp}(\xi)\) and \(\varphi|_{\text{supp}(\xi)}\) is proper (it is a hard task to find such a function).

Hence we can define \(\langle \varphi_* \xi, f \rangle := \langle \xi, \rho \cdot (f \circ \varphi) \rangle\). Note that \(\text{supp}(\rho \cdot (f \circ \varphi)) \subseteq \text{supp}(\rho) \cap \varphi^{-1}(\text{supp}(f)) \subseteq \varphi|_{\text{supp}(\rho)}^{-1}(\text{supp}(f))\). Since \(\varphi|_{\text{supp}(\rho)}\) is proper, and \(f\) is
Proposition 8.1.3. Let \( X \) compactly supported, that is \( \text{Dens}(X) \) identify the space of smooth measures \( \mu_c^\infty(X) \) with the space of smooth sections of the density bundle \( C_c^\infty(X, \text{Dens}(X)) \). Note that we can define \( \varphi_* : C_c^\infty(X, \text{Dens}(X)) \to \text{Dist}(X) \) by \( \langle \varphi_*(\mu), f \rangle = \int_X f d\mu \).

Exercise 8.1.2. Let \( X \) and \( Y \) be either smooth or \( F \)-analytic manifolds and \( \varphi : X \to Y \) a map. Show that the pushforward of a compactly supported distribution is compactly supported, that is \( \varphi_*(\text{Dist}_c(X)) \subseteq \text{Dist}_c(Y) \).

Proposition 8.1.3. Let \( X \) and \( Y \) be either smooth or \( F \)-analytic manifolds and \( \varphi : X \to Y \) a map. Show that if \( \varphi : X \to Y \) is a submersion, then:

1. \( \varphi_*(\mu_c^\infty(X)) \subseteq \mu_c^\infty(Y) \).
2. In addition, if \( \phi = f \cdot |\omega_X| \) and \( \varphi_*(f \cdot |\omega_X|) = g \cdot |\omega_Y| \), where \( |\omega_X| \) and \( |\omega_Y| \) are non-vanishing densities on \( X \) and \( Y \) respectively then

\[
g(y) = \int_{\varphi^{-1}(y)} f \frac{|\omega_X|}{|\varphi_*\omega_Y|}
\]

where \( \frac{|\omega_X|}{|\varphi_*\omega_Y|} \in \text{Dens}(\varphi^{-1}(y)) \) satisfies that \( |\omega_X| \) is sent to \( \frac{|\omega_X|}{|\varphi_*\omega_Y|} \otimes |\omega_Y| \) via the natural isomorphism \( \text{Dens}(X)_x \simeq \text{Dens}(\varphi^{-1}(y))_x \otimes \text{Dens}(Y)_{\varphi(x)} \).

Proof.

1. We prove the first statement in two steps. Case 1: \( X = F^n, Y = F^m \) where \( n \geq m \) and \( \varphi : F^n \to F^m \) is the natural projection \( \varphi(x_1, \ldots, x_n) = x_1, \ldots, x_m \). Recall that \( \text{Haar}(X) \simeq \text{Haar}(Y) \otimes \text{Haar}(X/Y) \) or equivalently that

\[
\text{Dens}(X) \simeq \text{Dens}(Y) \otimes \text{Dens}(X/Y).
\]

Let \( \phi \in C_c^\infty(X, \text{Dens}(X)) \) and note that \( \phi = f \cdot d\mu_X \) where \( f \in C_c^\infty(X) \) and \( \mu_X \) is a normalized Haar measure (taking the value 1 on the unit ball of \( X = F^n \)), so we can write \( \mu_X = \mu_Y \otimes \mu_{X/Y} \). By definition, for any \( g \in C_c^\infty(Y) \) we have:

\[
\langle \varphi_*(\phi), g \rangle = \langle \phi, g \circ \varphi \rangle = \int_X f \cdot (g \circ \varphi) d\mu_X = \int_Y \int_{X/Y} f \cdot (g \circ \varphi) d\mu_Y \otimes \mu_{X/Y}.
\]

It is compactly supported, and since \( g \circ \varphi(x_1, \ldots, x_n) = g(x_1, \ldots, x_m) \) depends only on \( Y \) we have that

\[
\langle \varphi_*(\phi), g \rangle = \int_Y \left( \int_{X/Y} f \cdot d\mu_{X/Y} \right) \cdot gd\mu_Y = \int_Y f \cdot gd\mu_Y
\]

where \( f \in C_c^\infty(Y) \). Hence \( \varphi_*(\phi) \) is a smooth measure.

Case 2 - general case: Let \( \varphi : X \to Y \) be a submersion. Take trivializing covers \( Y = \bigcup_{j \in J} V_j \) and \( X = \bigcup_{i \in I} U_i \) such that \( \varphi(U_i) \subseteq V_j \). For any \( i, j \) such
that $\varphi(U_i) \subseteq V_j$ we can choose isomorphisms $\tau_j : U_i \cong F^n$ and $\psi_j : V_j \cong F^m$ (if $X$ and $Y$ are $F$-analytic we choose isomorphisms to $\mathcal{O}_F$ to some powers) such that $\psi_j \circ \varphi \circ \tau_j^{-1}$ is the natural projection $F^n \to F^m$ (respectively $\mathcal{O}_F^n \to \mathcal{O}_F^m$ for $F$-analytic). Hence $(\psi_j \circ \varphi \circ \tau_j^{-1})_\ast (\mu^\infty_c(F^n)) \subseteq (\mu^\infty_c(F^m))$ and $\varphi_\ast(C^\infty_c(U_i, \text{Dens}(U_i))) \subseteq C^\infty_c(V_j, \text{Dens}(V_j)).$

Now, let $\phi \in C^\infty_c(X, \text{Dens}(X))$. Using partition of unity, we can write $\phi = \sum_{i \in I} f_i \mu_i$ where $f_i \mu_i \in C^\infty_c(U_i, \text{Dens}(U_i))$. Note that this is a finite sum since $\phi$ is compactly supported and observe that:

$$\varphi_\ast(\phi) = \varphi_\ast \left( \sum_{i \in I} f_i \mu_i \right) = \sum_{i \in I} \varphi_\ast(f_i \mu_i) = \sum_{i \in I} g_i \mu_i'$$

where $g_i \in C^\infty_c(V_j, \text{Dens}(V_j))$. Each $g_i \mu_i'$ is a smooth compactly supported measure, so the sum $\sum_{i \in I} g_i \mu_i'$ is a smooth section of the density bundle and we are done.

(2) Since $\varphi$ is a submersion for any $\varphi(x) = y \in Y$ we have that $\varphi^{-1}(y)$ is a submanifold of $X$ and the following exact sequence:

$$0 \to T_x\varphi^{-1}(y) \to T_x(X) \to T_{\varphi(x)}(Y) \to 0.$$  

Since this is an exact sequence of vector spaces it splits so $T_x(X) = T_x\varphi^{-1}(y) \oplus T_{\varphi(x)}(Y)$ and by dualizing we get that

$$T^\ast_x(X) = T^\ast_x\varphi^{-1}(y) \oplus T^\ast_{\varphi(x)}(Y).$$  

This implies that $\text{Dens}(X)_x = \text{Dens}(\varphi^{-1}(y))_x \otimes (\text{Dens}(Y))_{\varphi(x)}$.

We now reduce the problem to a small neighborhood. As before take trivializing covers $Y = \bigcup_{j \in J} V_j$ and $X = \bigcup_{i \in I} U_i$ such that $\varphi(U_i) \subseteq V_j$, and choose appropriate isomorphisms $\tau_i$ and $\psi_j$ for $\varphi(U_i) \subseteq V_j$ such that $\psi_j \circ \varphi \circ \tau_i^{-1}$ is the natural projection $F^n \to F^m$ (resp. $\mathcal{O}_F^n \to \mathcal{O}_F^m$).

We need to prove that for every $h \in C^\infty_c(Y)$ we have that

$$\langle \varphi_\ast(f |\omega_X|), h \rangle = \langle f |\omega_X|, h \circ \varphi \rangle = \langle g \cdot |\omega_Y|, h \rangle,$$

where $g$ is as in the statement of the proposition. Construct a partition of unity $f = \sum_{i \in I} f_i$ with respect to $\{U_i\}_{i \in I}$. Then it is enough to prove the claim for $f_i |\omega_X|$ as then:

$$\varphi_\ast(f |\omega_X|)(h) = \varphi_\ast(\sum_{i \in I} f_i |\omega_X|)(h) = \sum_{i \in I} \int_Y g_i h |\omega_Y|$$

where $g_i(y) = \int_{\varphi^{-1}(y)} f_i \eta$ since $\text{supp}(f_i \eta) \subseteq U_i$. As $g = \sum_{i \in I} g_i$ we would have that $g(y) = \int_{\varphi^{-1}(y)} f \eta$ as required.

In (1) we showed the case where $\varphi$ is a projection. Using the fact that for diffeomorphisms pushforward and pullback are inverse to one another
and using part (1) we get that:
\[
\psi_j \circ \varphi^* (f_i \omega_X) = \psi_j \circ \varphi \circ (\tau_i^{-1})_* (\omega_X)
\]
\[
= \psi_j \circ \varphi \circ (\tau_i^{-1})_* (f_i \circ \tau_i^{-1} \cdot (\tau_i^{-1})^* \omega_X) = \tilde{g}_i |(\psi_j^{-1})^* \omega_Y|
\]
where \(\tilde{g}_i(x) = \int_{\tau_i \circ \varphi^{-1}(x)} f_i \circ \tau_i^{-1} \left| (\tau_i^{-1})^* \omega_X \right| \). Set \(g_i := \tilde{g}_i \circ \psi_j\), then
\[
g_i(y) = \tilde{g}_i(\psi_j(y)) = \int_{\tau_i \circ \varphi^{-1}(y)} f_i \circ \tau_i^{-1} \left| (\tau_i^{-1})^* \omega_X \right| = \int_{\varphi^{-1}(y)} f_i \left| \frac{\omega_X}{\varphi^* \omega_Y} \right|.
\]

\[\square\]

**Definition 8.1.4.** By the proposition, the map \(\varphi_* : C^\infty_c(X, \text{Dens}(X)) \to C^\infty_c(Y, \text{Dens}(Y))\) gives rise to a pullback \(\varphi^* : C^{-\infty}(Y) \to C^{-\infty}(X)\).

**Exercise 8.1.5.** Let \(\varphi : X \to Y\) be a submersion. We can define a pullback \(\varphi^* : C^\infty(Y) \to C^\infty(X)\) both by \(\varphi^*(f) = f \circ \varphi\) and by the restriction of the map \(\varphi^* : C^{-\infty}(Y) \to C^{-\infty}(X)\) to the subspace \(C^\infty(Y) \subset C^{-\infty}(Y)\). Show that these two definitions coincide.

We can generalize the pushforward and pullback operations on functions and on distributions to functions and distributions with values in a vector bundle:

**Definition 8.1.6.** Let \(\varphi : X \to Y\) and let be \(\pi : E \to Y\) a bundle.

1. Define the pullback of the bundle \((E, \pi)\) to \(X\) by \(\varphi^* (E) := \{(x, e) \in X \times E : \varphi(x) = \pi(e)\}\) with the natural projection to \(X\).
2. Using (1) define the the pullback of sections \(\varphi^* : C^\infty(Y, E) \to C^\infty(X, \varphi^* (E))\) and by dualizing the pushforward of distributions \(\varphi_* : \text{Dist}(X, \varphi^* (E))_{\text{prop}} \to \text{Dist}(Y, E)\).
3. Define \(\varphi^! (E) := \varphi^* (E) \otimes \varphi^* (\text{Dens}(Y)^*) \otimes \text{Dens}(X)\).

**Proposition 8.1.7.** Let \(\varphi : X \to Y\) be a submersion. Then
\[
\varphi_* (C^\infty_c(X, \varphi^* (E) \otimes \text{Dens}(X))) \subseteq C^\infty(Y, E \otimes \text{Dens}(Y)).
\]
In particular, this implies that \(\varphi_* (C^\infty_c(X, \varphi^! (E))) \subseteq C^\infty(Y, E)\).

**Proof.** As in the proof of the last proposition, we may reduce to the case where \(\varphi : X \to Y\) is the natural projection, and \(X = F^n, Y = F^m\), and \(E \simeq F^n \times F^k\) is trivial (resp. \(O^F_F, O^F_P\) and \(O^P_F \times O^F_P\) for \(F\)-analytic manifolds). As a consequence, \(\varphi^* (E) = F^n \times F^k\) (resp. \(O^F_F \times O^F_P\)). Note reducing is possible since the notion of smoothness of a distribution (that is, it is a smooth measure) is local.
Let \( \phi = f \mu \in C_c^\infty(X, \varphi^*(E) \otimes \text{Dens}(X)) \). Then we have for any \( g \in C^\infty(Y, E) \),

\[
\langle \varphi_* (\phi), g \rangle = \langle \phi, g \circ \varphi \rangle = \int_X f \cdot (g \circ \varphi) \mu_X = \int_Y \left( \int_{X/Y} f \cdot \mu_{X/Y} \right) \cdot g \mu_Y
\]

\[
= \int_Y \tilde{f} \cdot g \mu_Y = \langle \tilde{f} \mu_Y, g \rangle,
\]

where \( \tilde{f} = \int_{X/Y} f \cdot \mu_{X/Y} \) which is smooth, so \( \varphi_* (\phi) \) is smooth. \( \Box \)

9. Lecture 9: Fourier Transform

**Definition 9.0.1.** Let \( G \) be a locally compact Hausdorff abelian group. Define its Pontryagin dual by,

\[
G' = \{ \chi : G \to U_1(\mathbb{C}) = S^1 \subseteq \mathbb{C} | \chi(g_1g_2) = \chi(g_1)\chi(g_2), \chi \text{ is cts} \}.
\]

The topology on \( G' \) is the compact open topology, i.e. a sub-basis of the topology is comprised of sets \( M(K,V) = \{ \chi \in G' : \chi(K) \subseteq V \} \) where \( K \subseteq G \) is compact and \( V \subseteq S^1 \) is open.

**Theorem 9.0.2.** Let \( G \) be a locally compact, Hausdorff abelian group, then \( G' \) is a locally compact Hausdorff abelian group.

**Proof.** We see that characters form an abelian group. Since \( S^1 \) is a topological group, the compact open topology on \( G' \) is equivalent to the topology of uniform convergence on compact sets. Thus, in order to show that the multiplication and inverse operations are continuous, it is enough to show that if \( f_n \to f \) and \( g_n \to g \) uniformly on compact sets then \( f_n \cdot g_n^{-1} \to f \cdot g^{-1} \) uniformly on compact sets. Now, if \( K \subset G \) is compact, note that this follows from the following bound \( (\forall x \in K) \):

\[
|f_n g_n^{-1} - f g^{-1}| \leq |f_n (g_n^{-1} - g^{-1})| + |(f_n - f) g^{-1}| = |g_n^{-1} - g| + |f_n - f|.
\]

Now to show it is locally compact, consider the space \((S^1)^G\) of all functions \( f : G \to S^1 \simeq \mathbb{R}/\mathbb{Z} \) with the product topology (i.e. a basis is given by open sets in only finitely many components). It is a compact space by Tychonoff’s theorem, and it has the space

\[
\hat{G} = \bigcap_{g_1, g_2 \in G} \{ \chi : G \to S^1 : \chi(g_1 g_2) = \chi(g_1) \chi(g_2) \},
\]

as a closed subspace, implying that \( \hat{G} \) is compact. Furthermore, for every \( S \subseteq G \) and \( \epsilon > 0 \) the set \( A(S, \epsilon) = \{ \chi \in (S^1)^G : \chi(S) \subseteq [-\epsilon, \epsilon] \} \) is also closed and compact in \((S^1)^G\) as the complement is a union of sets of the form

\[
\{ \chi : G \to S^1 : \exists s \in S \text{ s.t. } \chi(s) \in [-\epsilon, \epsilon] \}
\]

which are open.

In particular, taking an open neighborhood \( \epsilon \in U \subset G \), the sets \( V(U, \epsilon) = A(U, \epsilon) \cap \hat{G} \) are closed and compact in \((S^1)^G\). Take \( 0 < \epsilon < \frac{1}{2} \), we show that we have that
$V(U, \epsilon) \subseteq G^\vee$. Start with an open $e \in U_0 = U \subset G$, and choose a sequence of neighborhoods $(U_n)_{n=1}^\infty$ in $G$ such that $U_{n+1} \cdot U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$ and set $\epsilon_n = \frac{\epsilon}{2n}$. Taking $\chi \in V(U_n, \epsilon_n)$, we see that since for $x \in U_{n+1}$ we have that $\chi(x) \in [-\epsilon_n, \epsilon_n]$ and $x^2 \in U_n$ we get $\chi(x^2) = \chi(x)^2 \in [-\epsilon_n^2, \epsilon_n^2] \subseteq [-\frac{\epsilon_n}{2}, \frac{\epsilon_n}{2}]$, implying that $V(U_n, \epsilon_n) \subseteq V(U_{n+1}, \epsilon_{n+1})$.

Now, take $\chi \in V(U, \epsilon)$ and a basic open set $(-\delta, \delta) \subset S^1$ for $\delta > 0$. We have that $[-\epsilon_n, \epsilon_n] \subseteq (-\delta, \delta)$ for $n$ big enough, implying that $e \in U_n \subseteq \chi^{-1}((-\delta, \delta))$ which means that $\chi$ is continuous at $e$. Since $\chi$ is a homomorphism, we can show it is continuous everywhere; if $\chi(g) \in W \subset S^1$ and $W$ is open, we have that $(-\delta, \delta) \subseteq \chi(g^{-1})W$ for some $\delta > 0$ and that,

$$\chi^{-1}(\chi(g^{-1})W) = \{y \in G : \chi(y) \in \chi^{-1}(g)W\} = \{y \in G : \chi(\chi(y)) \in W\} = g^{-1}\chi^{-1}(W).$$

Now, for some $m \in \mathbb{N}_0$ big enough, the following implies that $g \in gU_m \subseteq \chi^{-1}(W)$:

$$U_m \subset \chi^{-1}(\chi(g^{-1})W) = g^{-1}\chi^{-1}(W).$$

We know that $V(U, \epsilon)$ is compact in the product topology, and want to show it is compact with respect to the compact open topology. For this, it is enough to show that any net in $V(U, \epsilon)$ has a converging subnet in the compact open topology. Assume we are given some net $(x_\alpha) \in V(U, \epsilon)$, then it has a subnet $(f_\beta) \to f$ converging in the product topology with $f_\beta, f \in V(U, \epsilon)$. Now, note that $V(U, \epsilon)$ is uniformly equicontinuous, that is if $g_1, g_2 \in G$ and $g_1 g_2^{-1} \in U_n$ then for any $\chi \in V(U, \epsilon),$

$$|\chi(g_1) - \chi(g_2)| = |\chi(g_1)\chi^{-1}(g_2) - 1| = |\chi(g_1 g_2^{-1}) - 1| \leq \epsilon_n.$$ 

Given a basic open neighborhood of the identity character $1_G \in M(K, B_\nu(0))$, where $K$ is compact, for every $g \in K$ we have that $g \in U_n g$ (for $n$ big enough). Now, taking any $g' \in U_n g$, we get that $g'g^{-1} \in U_n$ implying that for some big enough $\beta$ we have that $|f(g) - f_\beta(g)| < \epsilon_n$ and that,

$$|f(g') - f_\beta(g')| \leq |f(g') - f(g)| + |f_\beta(g') - f_\beta(g)| + |f(g) - f_\beta(g)| < 3\epsilon_n.$$

Taking $n > n_q$ such that $\epsilon_n < \frac{\epsilon}{2^k}$, we see that $f_\beta \to f$ uniformly on $U_n g$, but since $K$ is compact we can cover it with finitely many sets of the form $U_{n_q} g$, and take $n = \max \{n_q \}$ and appropriate $\beta$.

To finish off the argument, note that by local compactness every $g \in G$ has a neighborhood $g \in U$ with compact closure $U \subset K$, and we have that $M(K, B_\nu(0)) \subseteq V(U, \epsilon)$ for an appropriate $\frac{1}{2} > \epsilon > 0$. □

**Exercise 9.0.3.** Let $G$ be a locally compact, Hausdorff abelian group. Show that if $G$ is compact then $G^\vee$ is discrete, and that if $G$ is discrete then $G^\vee$ is compact.
Theorem 9.0.4. For a locally compact abelian group $G$, we have that the natural map $\varphi : G \to G^\vee$ defined by $g \mapsto \varphi_g$, where $\varphi_g(\chi) = \chi(g)$, is an isomorphism $G^\vee \simeq G$.

Proof. This is complicated. Need a reference. \hfill \Box

Exercise 9.0.5. Let $G$ be a locally compact, Hausdorff abelian group, and $H \leq G$ a closed subgroup. Show that:

(1) Pontryagin duality is a contravariant endofunctor in the category of locally compact abelian groups.

(2) $H^\vee \simeq G^\vee / H^\perp$ where $H^\perp = \{ \chi \in G^\vee : \chi(h) = 1 \ \forall h \in H \}$, and that if $H$ and $G$ are vector spaces then this is a homeomorphism (Hint: use an appropriate version of the Hahn-Banach theorem).

Example 9.0.6.

(1) For any finite abelian group $G$ we have that $G \simeq G^\vee$.

(2) The dual of $U_1(\mathbb{C}) = S^1$ is $\mathbb{Z}$.

(3) We have that $\mathbb{R}^\vee \simeq \mathbb{R}$.

Exercise 9.0.7. Let $V$ be a topological vector space over a local field $F$. Then $V^* \otimes_F F^\vee \simeq V^\vee$.

Definition 9.0.8. Let $G$ be a locally compact Hausdorff abelian group. The map $F : \mu_c(G) \to C(G^\vee)$ defined by $F(\mu)(\chi) = \int \chi d\mu$ is called the Fourier transform.

Exercise 9.0.9. Show the following:

(1) $F$ is continuous.

(2) Let $G$ be a locally compact abelian group. For a character $\tau : G \to S^1$ define $sh_\tau(x) = \tau(x + h)$. Show that for $\eta \in \mu^\infty_c(G)$ and $g \in G$:

(a) $F(sh_\tau(\eta))(\chi) = \chi(g)F(\eta)(\chi)$ for all $\chi \in G^\vee$.

(b) $F(\chi \eta) = sh_{\chi^{-1}}(F(\eta))$ for all $\chi \in G^\vee$.

Definition 9.0.10. Let $X_1$ and $X_2$ be locally compact topological vector spaces and let $\mu_1 \in \mu_c^\infty(X_1)$ and $\mu_2 \in \mu_c^\infty(X_2)$. We define the external tensor product of such measures $\mu_1 \otimes \mu_2 \in \mu_c^\infty(X_1 \times X_2)$. In addition, If $X_1 = X_2 = G$, then we define the convolution of these measures by $\mu_1 * \mu_2 := m_*(\mu_1 \otimes \mu_2)$ where $m : G \times G \to G$ is the multiplication map.

Fact 9.0.11. For two measures $\alpha, \beta \in \mu_c^\infty(G)$ we have that $F(\alpha * \beta) = F(\alpha) \cdot F(\beta)$.

Definition 9.0.12. Let $V$ be a finite dimensional vector space over a local field $F$. Define the space of Schwartz functions $S(V)$ on $V$ by:

(1) If $F$ is non-archimedean, then $S(V) = C_c^\infty(V)$, i.e. locally constant functions on $V$. 
Definition 9.0.14. □

The proof for vector spaces over non-Archimedean fields is analogous.

Remark that since the functions $F$ map Banach spaces into Banach spaces, the other side of the coin is given by integration by parts, where one can differentiate directly using the definition to verify the above procedure. The other side of the coin is given by integration by parts.

Proposition 9.0.13. The Fourier transform $F : S(V,\text{Haar}(V)) \to S(V^\vee)$ is continuous for an Archimedean $V$ and its image is indeed contained in $S(V^\vee)$ in both cases.

Proof. Assume $V$ is a real vector space of dimension $n$, and recall that the topology on $S(V)$ is determined by the semi-norms $\|f\|_{\alpha,\beta} = \sup_{x \in V} |\Phi_\alpha(x) \frac{\partial^\beta f(x)}{\partial x^\beta}|$ where $\alpha, \beta \in \mathbb{N}_0^n$ and $\Phi_\alpha(x) = \prod_{j=1}^{n} x_j^{\alpha_j}$. It is enough to show that for every $f \in C_c^\infty(V,\text{Haar}(V))$ and semi-norm $\|\cdot\|_{\alpha,\beta}$ on $S(V^\vee)$ there exists a semi-norm $\|\cdot\|'$ on $S(V,\text{Haar}(V))$ and a positive constant $C$ such that $\|F(f)\|_{\alpha,\beta} \leq C\|f\|'$. Now, recall that,

$$i\partial F(f) \frac{\partial}{\partial \xi_j} = \int_{\mathbb{R}^n} i\partial \Phi_\xi \left( e^{-i\xi \cdot x} f(x) \right) dx = F(x_j f),$$

where one can differentiate directly using the definition to verify the above procedure. The other side of the coin is given by integration by parts,

$$\xi_j F(f) = \int_{\mathbb{R}^n} \xi_j e^{-i\xi \cdot x} f(x) dx = \left[-e^{-i\xi \cdot x} f(x)\right]_{-\infty}^{\infty} - \int_{\mathbb{R}^n} \xi_j e^{-i\xi \cdot x} \frac{\partial f(x)}{\partial x_j} dx = F(-i\partial f).$$

Note that since the functions $e^{-i\xi \cdot x}$ converge weakly to zero as distributions as $|\xi| \to \infty$ we get that Schwartz measures are mapped into $S(V^\vee)$. We can now bound $F(f)$ properly using the above relations:

$$\|F(f)\|_{\alpha,\beta} = \sup_{x \in V^\vee} \left| \Phi_\alpha(x) \frac{\partial^\beta F(f)(x)}{\partial (x^\vee)^\beta} \right| \leq \sup_{x \in V^\vee} \left| x^\vee \frac{\partial^\beta (\Phi_\beta(-x) f)}{\partial x^\alpha} \right| \mu \leq C \sup_{x \in V} \left( 1 + |x| \right)^{n+1} \left| \frac{\partial^\beta (\Phi_\beta(|x|) f)}{\partial x^\alpha} \right| d\mu(x).$$

where $C = \int_{V} \frac{1}{(1+|x|)^{n+1}} d\mu(x)$. Since the last expression is a linear combination of norms of the form $\|f\|_{\alpha',\beta'}$ for $|\alpha'| \leq |\alpha| + n + 1$ and $|\beta'| \leq |\beta|$, this implies that $F$ is continuous. Note that we can also use this to show that $F(f)$ is Schwartz, since if all the norms $\|\cdot\|_{\alpha,\beta}$ are bounded then the value of $|\Phi_\alpha(x) \frac{\partial^\beta f(x)}{\partial x^\beta}|$ decays to 0 as $|x| \to \infty$ for every $\alpha$ and $\beta$.

The proof for vector spaces over non-Archimedean fields is analogous. □

Definition 9.0.14. Let $S(V)$ be the space of Schwartz functions on $V$.

(1) We call $\xi \in S^\ast(V)$ the space of tempered distributions and $G(V) := S^\ast(V,\text{Haar}(V))$ the space of tempered generalized functions.
(2) Finally, we define the Fourier transform on tempered distributions via duality:
\[
\mathcal{F}^* : S^*(V^\vee) \to \mathcal{G}(V) := S^*(V, \text{Haar}(V)).
\]
Taking \(V := V^\vee\) we get \(\mathcal{F}^* : S^*(V) \to \mathcal{G}(V^\vee)\).

**Theorem 9.0.15.** The definition of Fourier transform of distributions is consistent with the definition given for functions. In other words \(\mathcal{F}^*|_{\mathcal{S}(V, \text{Haar}(V))} = \mathcal{F}\).

**Proof.** Let \(f(x) \cdot dx \in \mathcal{S}(V, \text{Haar}(V))\) and \(g(\chi) \cdot d\chi \in \mathcal{S}(V^\vee, \text{Haar}(V^\vee))\). Then by definition,
\[
\langle \mathcal{F}^*(f(x) \cdot dx), g(\chi) \cdot d\chi \rangle := \langle f(x) \cdot dx, \mathcal{F}(g(\chi) d\chi) \rangle = \int_V f(x) \mathcal{F}(g(\chi) d\chi)(x) dx
\]
where \(\mathcal{F}(g(\chi) d\chi)(x) := \int_{V^\vee} \chi(x) g(\chi) d\chi\). Therefore we have:
\[
\int_V f(x) \mathcal{F}(g(\chi) d\chi)(x) dx = \int_V f(x) \int_{V^\vee} \chi(x) g(\chi) d\chi dx = \int_{V^\vee} \left( \int_V \chi(x) f(x) dx \right) g(\chi) d\chi
\]
\[
= \int_{V^\vee} (\mathcal{F}(f)(\chi)) g(\chi) d\chi = \langle \mathcal{F}(f) \cdot dx, g(\chi) \cdot d\chi \rangle.
\]

**Remark 9.0.16.** We will usually omit the * from the \(\mathcal{F}^*\) notations, this should cause no confusion.

\[\square\]

In the following argument we would like to show the Fourier transform is a unitary operator. For this we will first need to define a pairing between \(\text{Haar}(V)\) and \(\text{Haar}(V^\vee)\). Given \(\alpha \in \text{Haar}(V)\) and \(\beta \in \text{Haar}(V^\vee)\) we can define such a pairing as follows. We choose \(f \in C_c^\infty(V^\vee)\) such that \(f(0) = 1\) and then define \(\langle \alpha, \beta \rangle := \langle \mathcal{F}(\alpha), f \cdot \beta \rangle\).

**Exercise 9.0.17.**

1. Show this is well defined. That is, given a different \(g \in C_c^\infty(V^\vee)\) such that \(g(0) = 1\), show that \(\langle \mathcal{F}(\alpha), (f - g) \cdot \beta \rangle = 0\).

2. Show that \(\text{Haar}(V^\vee) \simeq_{\text{can}} \text{Haar}(V)^*\).

**Definition 9.0.18.** We define a map \(\mathcal{F}_n : S^*(V, \text{Haar}(V)^\otimes n) \to S^*(V^\vee, \text{Haar}(V^\vee)^\otimes (1-n))\) such that \(\mathcal{F}_n\) is the Fourier transform. We use the following isomorphisms:

1. The pairing \(\text{Haar}(V^\vee) \simeq_{\text{can}} \text{Haar}(V)^*\).

2. The identification \(S^*(V, \text{Haar}(V)^\otimes n) \simeq S^*(V) \otimes \text{Haar}(V)^\otimes -n\).

3. The identification \(S^*(V^\vee, \text{Haar}(V^\vee)^\otimes (1-n)) \simeq S^*(V^\vee, \text{Haar}(V^\vee)) \otimes \text{Haar}(V^\vee)^\otimes n\).

The first item was shown in the previous exercise. The second identification is as follows. Given \(\xi \otimes \beta \in S^*(V) \otimes \text{Haar}(V^\vee)^\otimes n\) and \(f \cdot \alpha \in S(V, \text{Haar}(V)^\otimes n)\) we have that \(\langle \xi \otimes \beta, f \alpha \rangle = \langle \xi, f \rangle \langle \beta, \alpha \rangle\). The third identification is similar.
Finally, we define the map by applying the Fourier transform on the first coordinate of the right hand side of (2) and by applying the canonical map (1) on the second coordinate.

**Proposition 9.0.19.** We have that $\mathcal{F}_1 \circ \mathcal{F}_0 = \text{flip}$ where $(\text{flip}(\xi), f(x)\mu) = (\xi, f(-x)\mu)$.

**Proof.** Note that $\text{span}\{\delta_x\}_{x \in V}$ is a dense subspace of $S^*(V)$ with respect to the weak topology. Hence it is enough to show that $\mathcal{F}_1 \circ \mathcal{F}_0(\delta_a) = \delta_{-a}$ for all $a \in V$. Note that $\langle \mathcal{F}_0(\delta_0), f\beta \rangle := \langle \delta_0, \mathcal{F}_0(f\beta) \rangle = \int_{V^\vee} f \, d\beta$, this implies $\mathcal{F}_0(\delta_0) = 1$.

As before, $\mathcal{F}_1 : S^* (V^\vee, \text{Haar}(V^\vee)) \to S^* (V)$ is defined by identifying $S^* (V^\vee, \text{Haar}(V^\vee))$ with $S^* (V^\vee) \otimes \text{Haar}(V)$. Under this identification, $1 \cdot \mu$ for a choice of a Haar measure $\mu$ on $V^\vee$ is identified with $1 \otimes \eta$ where $\eta \in \text{Haar}(V)$ and $\langle \mu, \eta \rangle = 1$.

Given $f \in S(V)$, we have that (note we are using Theorem 9.0.15):

$$\langle \mathcal{F}_1(1 \cdot \mu), f \rangle = \langle \mathcal{F}^*(1) \otimes \eta, f \cdot \eta \otimes \mu \rangle = \langle \mathcal{F}(1), f \cdot \eta \rangle \langle \eta, \mu \rangle = \langle \delta_0, f \cdot \eta \rangle \cdot 1 = f(0),$$

so $\mathcal{F}_1 \circ \mathcal{F}_0(\delta_0) = \delta_0$.

Using Exercise 9.0.9, we now see that (here $\chi$ is the function which substitutes the value $a$ in a given character $\chi$):

$$\mathcal{F}_1 \circ \mathcal{F}_0(\delta_a) = \mathcal{F}_1 \circ \mathcal{F}_0(\text{sh}_u(\delta_0)) = \mathcal{F}_1(\chi(a) \mathcal{F}_0(\delta_0)) = \text{sh}_{-a} \mathcal{F}_1 \circ \mathcal{F}_0(\delta_0) = \delta_{-a}.$$  

By continuity of $\mathcal{F}_0$ and $\mathcal{F}_1$ this implies that $\mathcal{F}_1 \circ \mathcal{F}_0 = \text{flip}$. □

**Definition 9.0.20.** Let $F$ and $K$ be local fields and $\chi : F^\times \to K^\times$ a character. For a 1-dimensional space $V$ over $F$ we define a functor by:

$$\chi(V) := \{ \varphi : V^* \to K : \varphi(\alpha v) = \chi(\alpha) \varphi(v) \forall \alpha \in F^\times, v \in V^* \}.$$

**Example 9.0.21.** Let $\chi : F^\times \to F^\times$ be the character $x \mapsto x^2$ and let $V$ be a one dimensional vector space of $F$. Then

$$\chi(V) := \{ \varphi : V^* \to K : \varphi(\alpha f) = \alpha^2 \varphi(f) \}.$$

Note that $\chi(V) \simeq_{\text{can}} V \otimes V$ by $v \otimes w \mapsto \varphi_v \cdot \varphi_w$. Indeed, given $\psi \in V^*$, we have $\varphi_v \cdot \varphi_w(\psi) = \psi(v) \cdot \psi(w)$ and $\varphi_v \cdot \varphi_w(\psi(\alpha)) = a \psi(v) \cdot a \psi(w) = a^2 \varphi_v \cdot \varphi_w(\psi)$.

**Definition 9.0.22.** Let $V$ be a 1-dimensional vector space over $\mathbb{R}$.

(1) A positive structure on $V$ is a non trivial subset $P \subseteq V$ such that $\mathbb{R}_{\geq 0} \cdot P = V$.

(2) If $V$ has a positive structure, we define

$$V^\alpha := |V|^\alpha = \{ \varphi : V^* \to \mathbb{R} : \varphi(\beta f) = |\beta|^\alpha \varphi(f) \}.$$

**Exercise 9.0.23.** Let $V$ be a real 1-dimensional vector space with a positive structure.

(1) Show that:

(a) $V \simeq_{\text{can}} |V|$. 

(b) \( V^{\alpha + \beta} \simeq_{\text{can}} V^\alpha \otimes V^\beta \) where \( \alpha, \beta \in \mathbb{Q}^\times \).

(2) Deduce that \( \text{Haar}(V)^\alpha \otimes \text{Haar}(V)^\beta \).

**Definition 9.0.24.** For \( \alpha \in \mathbb{Q} \) we define similarly to the procedure defined above,

\[ F_\alpha : S^*(V, \text{Haar}(V)) \to S^*(V^\vee, \text{Haar}(V^\vee)^{1-\alpha}). \]

In particular, choosing \( \alpha = \frac{1}{2} \) we have:

\[ F_{\frac{1}{2}} : S^*(V, \text{Haar}(V)^{\frac{1}{2}}) \to S^*(V^\vee, \text{Haar}(V^\vee)^{\frac{1}{2}}). \]

**Theorem 9.0.25 (Functoriality of Fourier transform).** Let \( W \subset V \) be vector spaces over a local field, denote the inclusion of \( W \) in \( V \) by \( i \), and set \( p : V^\vee \to W^\vee \) for the induced linear map on the duals, then the following diagrams commute:

\[
\begin{array}{ccc}
S(V) & \xrightarrow{i^*} & S(W) \\
\mathcal{F} & & \mathcal{F} \\
S(V^\vee, \text{Haar}(V^\vee)) & \xrightarrow{p^*} & S(W^\vee, \text{Haar}(W^\vee)) \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
S^*(V) & \xleftarrow{i_*} & S^*(W) \\
\mathcal{F} & & \mathcal{F} \\
G(V^\vee) & \xleftarrow{p^*} & G(W^\vee). \\
\end{array}
\]

Note that this is possible since \( p \) is a submersion (linear and surjective) so pushing Schwartz measures along it yields Schwartz measures.

**Proof.** We start by showing the right hand side diagram commutes. Since \( i_* \), the Fourier transform and \( p^* \) are continuous with respect to the weak topology, it is enough to prove commutativity for a dense set in \( S^*(W) \).

First take the delta function \( \delta_0 \in S^*(W) \), it is a compactly supported measure, and it holds that \( i_*(\delta_0) = \delta_0 \). Furthermore, since \( \mathcal{F} : S^*(V) \to G(V^\vee) \) is defined via duality we have that \( \mathcal{F}(\delta_0) = 1 \):

\[ \langle \mathcal{F}(\delta_0), f \mu \rangle = \langle \delta_0, \mathcal{F}(f \mu) \rangle = F(f \mu)(0_{V^\vee}) = \int_{V^\vee} f \mathrm{d} \mu = \langle 1, f \mu \rangle, \]

where the third equality is sensible since \( \mathcal{F}(f \mu) \in S(V^\vee^\vee) \) and \( 0_{V^\vee^\vee}(\chi) = 1 \) for all \( \chi \in V^\vee \). We can also show that \( p^*(1) = 1 \). Consider \( G(W^\vee) \) as a subspace of \( C^{-\infty}(W^\vee) \), there the generalized Schwartz function 1 is a smooth function, and note that the following diagram, where the horizontal arrows are the inclusions is commutative:

\[
\begin{array}{ccc}
G(V^\vee) & \xleftarrow{1} & C^{-\infty}(V^\vee) \\
\uparrow p^* & & \uparrow p^* \\
G(W^\vee) & \xleftarrow{1} & C^{-\infty}(W^\vee) \\
\end{array}
\]
Now, note that every measure $f \mu \in \mu^{\infty}(V^\vee)$ can be treated either as a functional on smooth functions (since it has compact support as a distribution), or as the parameter a generalized function takes values on. This is utilized in the second equality below to yield the required result:

$$\langle p_\ast^{C,-\infty}(1), f \mu \rangle = \langle 1, p_\ast(f \mu) \rangle = \langle f \mu, p_\ast^{C}\ast(1) \rangle = \langle f \mu, 1 \rangle = \langle 1, f \mu \rangle.$$ 

Note that since $p_\ast$ is a submersion pushing forward a compactly supported smooth measure along it yields a smooth measure.

Since $\delta_w$ for any $w \in W$ is just a translation of $\delta_0$ by $w$, its Fourier transform is $\mathcal{F}(\delta_w)(\chi) = \chi(w)$, and $i_\ast$ and $p_\ast$ are invariant to translations, the diagram is commutative for delta distributions. The space of delta distributions $\text{span}_\mathbb{C}\{\delta_w\}_{w \in W}$ is dense w.r.t the weak topology since for every function $f$ with $f(x_0) \neq 0$ we can take suitable $c \in \mathbb{C}$ such that $|\langle \xi - c\delta_x, f \rangle|$ is small as desired.

To see this implies the commutativity of the left diagram, it is enough to show that if $A^\ast = 0$ for $A^\ast : V_2^\ast \to V_1^\ast$ where $A^\ast$ is the dual map to the linear map $A : V_1 \to V_2$, then $A = 0$, and use this for $\mathcal{F} i_\ast - p_\ast \mathcal{F}$.

If $A^\ast = 0$, we have for every $\xi_2 \in V_2^\ast$ and $v_1 \in V_1$ that $0 = \langle A^\ast \xi_2, v_1 \rangle = \langle \xi_2, Av_1 \rangle$. If there exists $v_1 \in V_1$ such that $Av_1 \neq 0$, then we can define a non-zero linear functional $\xi : \text{span}_\mathbb{C}\{Av_1\} \to \mathbb{C}$ via $\langle \xi, Av_1 \rangle = 1$, and extend it to a non-zero continuous functional $\xi_2 \in V_2^\ast$ by the Hahn-Banach theorem. This yields a contradiction as

$$1 = \langle \xi_2, Av_1 \rangle = \langle A^\ast \xi_2, v_1 \rangle = \langle 0, v_1 \rangle = 0.$$

□

References

