GENERALIZED FUNCTIONS LECTURES

Contents

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0. Preface

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1. The space of generalized functions on $\mathbb{R}^n$

1.1. **Motivation.** One of the most basic and important examples of a generalized function is the Dirac delta function. The Dirac delta function on $\mathbb{R}$ at point t is usually denoted by δ_t, and while it is not a function, it can be intuitively described by $\delta_t(x) := \begin{cases} \infty & x = t \\ 0 & x \neq t \end{cases}$, and by satisfying the equality $\int\limits_{-\infty}^{\infty} \delta_t(x)dx = 1$. Notice that it also satisfies:

$$\int\limits_{-\infty}^{\infty} \delta_t(x)f(x)dx = f(t)\int\limits_{-\infty}^{\infty} \delta_t(x)dx = f(t).$$

The following are possible motivations for generalized functions:

- Many natural analytic operations, *e.g.* derivation and Fourier transform, are not defined for arbitrary functions. In many cases, the reason is that the result of the operation is in fact a generalized function.
- Every real function $f : \mathbb{R} \to \mathbb{R}$ can be constructed as an (ill-defined) sum of continuum indicator functions $f := \sum\limits_{t\in\mathbb{R}} f(t)\delta_t$.
- In general, solutions to differential equations, and even just derivatives of functions are not functions, but rather generalized function. Using the language of generalized functions allows one to rigorize such notions.
- Generalized functions are useful in physics. For example, the density of a point mass can be described by the Dirac delta function. This allows to give a mathematical formulation to physical theories that unify both mass distribution and point masses, by using mass distributions that are allowed to be generalized functions.
- As a mathematical manifestation of the previous point, distributions can be used in series theory, since sequences can be replaced by discrete combinations of δ-functions, and thus summation of series is integration of the corresponding generalized function, and one can unify techniques of integration with techniques for series summation. For example, summation by parts becomes a special case of integration by parts, and Fourier series become a special case of Fourier Transform.

1.2. **Basic definitions.** In this book we will consider various spaces of functions and functionals. Unless specified otherwise, all the functions and functionals will be real-valued. All the statements below are also valid for complex-valued functions. In order to define what is a generalized function we first need to introduce some standard notation.

Definition 1.2.1 (Smooth functions of compact support)**.**

(i) *Denote by $C^\infty(\mathbb{R})$ the space of smooth functions $f:\mathbb{R}\to\mathbb{R}$, i.e. functions that can be differentiated infinitely many times.*
(ii) *Define the support of a function $f:\mathbb{R}\to\mathbb{R}$ by*

$$\operatorname{supp}(f) := \overline{\{x\in\mathbb{R} : f(x)\neq 0\}},$$

the closure of the set in which it does not vanish.
(iii) *Denote by $C_c^\infty(\mathbb{R})\subset C^\infty(\mathbb{R})$ the space of smooth functions with compact support.*

Definition 1.2.2 (Convergence in $C_c^\infty(\mathbb{R})$)**.** *Given $f\in C_c^\infty(\mathbb{R})$ and a sequence $\{f_n\}_{n=1}^\infty$ of smooth functions with compact support we say that $\{f_n\}_{n=1}^\infty$ converges to f in $C_c^\infty(\mathbb{R})$ if:*

(i) *There exists a compact set $K\subset\mathbb{R}$ for which $\bigcup\limits_{n\in\mathbb{N}}\operatorname{supp}(f_n)\subseteq K$.*
(ii) *For every order $k\in\mathbb{N}_0$, the derivatives $(f_n^{(k)})_{n=1}^\infty$ converge uniformly to the derivative $f^{(k)}$.*

Exercise 1.2.3. *Fix a non-zero $\rho\in C_c^\infty(\mathbb{R})$. Show that*

(i) *the sequence*

$$f_n(x) := \rho(x)\left(1+\frac{x}{n}\right)^n$$

converges in $C_c^\infty(\mathbb{R})$ to $\rho(x)e^x$.
(ii) *the sequence*

$$g_n(x) := \frac{\rho(x+n)}{n}$$

does not converge in $C_c^\infty(\mathbb{R})$, even though it satisfies condition (ii) of the definition.
(iii) *the sequence*

$$h_n(x) := \rho(x)\frac{\sin(2^n x)}{n}$$

does not converge in $C_c^\infty(\mathbb{R})$, even though it satisfies condition (i) of the definition, and converges uniformly to zero.

We can now define the notion of distributions (cf. [**?**, ??] and [Kan04, Section 2.3].

Definition 1.2.4 (Distributions)**.** *A linear functional $\xi: C_c^\infty(\mathbb{R})\to\mathbb{R}$ is* continuous *if for every convergent sequence $\{f_m\}_{m=1}^\infty$ of functions $f_m\in C_c^\infty(\mathbb{R})$ we have*

$$\lim_{m\to\infty}\langle\xi, f_m\rangle = \langle\xi, \lim_{m\to\infty} f_m\rangle.$$

We will usually use the notation $\langle\xi, f\rangle$ instead of $\xi(f)$. We call a continuous linear functional a distribution *or a* generalized function*. The space of all generalized functions on $\mathbb{R}$ is denoted by $C^{-\infty}(\mathbb{R}) := (C_c^\infty(\mathbb{R}))^*$.*

Remark 1.2.5. *In §?? below we will define a natural topology on the space $C_c^\infty(\mathbb{R})$. The convergence in this topology will be as in Definition 1.2.2, but this does not* *define the topology uniquely since this topology is not first countable. We will show that a linear functional on this topological space is continuous if and only if it satisfies the condition in Definition 1.2.4.*

Remark 1.2.6. *For now the names generalized functions and distributions are synonymous as there is no difference for $\mathbb{R}$. We will discuss the difference in a later part of the manuscript, when it will be relevant.*

Example 1.2.7. *For any $a \in \mathbb{R}$, define $\delta_a \in C^{-\infty}(\mathbb{R})$ by $\langle \delta_a, f\rangle := f(a)$.*

Recall that a function f is *locally-L^1* if the restriction to any compact subset in its domain is an L^1 function. We denote the space of such functions L^1_{Loc}. Given a real-valued function $f \in L^1_{\text{Loc}}(\mathbb{R})$ we define $\xi_f : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ to be the generalized function

$$\langle \xi_f, \phi\rangle := \int_{-\infty}^{\infty} f(x) \cdot \phi(x) dx.$$

Note that this integral converges as ϕ vanishes outside of some compact set K, and $(f\phi)|_K \in L^1(K)$. These are sometimes called *regular generalized functions.*

Exercise 1.2.8. *For any $f \in L^1_{\text{Loc}}(\mathbb{R})$, show that ξ_f is a well defined distribution.*

Note that we have

$$C^\infty(\mathbb{R}) \subset C(\mathbb{R}) \subset L^1_{\text{Loc}}(\mathbb{R}) \subset C^{-\infty}(\mathbb{R}),$$

where the last embedding is given by $f \mapsto \xi_f$. This embedding motivates the name generalized function.

Exercise 1.2.9. *Prove that there exists a function $f \in C_c^\infty(\mathbb{R})$ which is not the zero function.* ***Hint:*** *Use functions such as $e^{-1/(1-x)^2}$.*

Definition 1.2.10. *We say that a sequence of generalized functions $\{\xi_n\}_{n=1}^\infty$* weakly converges *to $\xi \in C^{-\infty}(\mathbb{R})$ if for every $f \in C_c^\infty(\mathbb{R})$ we have*

$$\lim_{n\to\infty} \langle \xi_n, f\rangle = \langle \xi, f\rangle.$$

Note that in particular this definition applies to locally-L^1 functions, since as we have seen above they are contained in the space of generalized functions. Now we can give an equivalent definition of the space of generalized functions - as the sequential completion of $C_c^\infty(\mathbb{R})$ with respect to the weak convergence. For this we need the notion of a weakly Cauchy sequence:

Definition 1.2.11. *(i) A sequence $\{f_n\}$ in $L^1_{\text{Loc}}(\mathbb{R})$ is called a* weakly Cauchy sequence *if for every $g \in C_c^\infty(\mathbb{R})$ and $\epsilon > 0$ there exists a number $N \in \mathbb{N}$ such that for all $m, n > N$ we have that $\left|\int_{-\infty}^{\infty} (f_n(x) - f_m(x))g(x)dx\right| < \epsilon$.*

(ii) Two weakly Cauchy sequences are called equivalent if their difference weakly converges to zero.

One can similarly define these notions for sequences of distributions.

Exercise 1.2.12. *A sequence $\{f_n\}$ in $C^{-\infty}(\mathbb{R})$ is a weakly Cauchy sequence if and only if for any $g \in C_c^\infty(\mathbb{R})$, the sequence $\langle f_n, g\rangle$ converges.*

However, weakly Cauchy sequences in $C_c^\infty(\mathbb{R})$ do not necessarily converge in $C_c^\infty(\mathbb{R})$.

Remark 1.2.13. *One can define the space of generalized functions $C^{-\infty}(\mathbb{R})$ as the space of equivalence classes of weakly Cauchy sequences in $C_c^\infty(\mathbb{R})$. As we will show in ??, this definition is equivalent to Definition 1.2.4. It is important that we take weakly Cauchy sequences rather than weakly Cauchy nets, since otherwise we would get the full completion of $C_c^\infty(\mathbb{R})$, which is larger than $C^{-\infty}(\mathbb{R})$, as we will see in ??.*

Exercise 1.2.14. *Find a sequence of functions $(f_n)_{n=1}^\infty$ in $C_c^\infty(\mathbb{R})$ converging weakly to 0 that does not converge point-wise.*

One can find a weakly Cauchy sequence that converges to the Dirac's delta.

Definition 1.2.15. *A sequence $\phi_n \in C_c(\mathbb{R})$ of continuous, non-negative, compactly supported functions is said to be an* approximation of identity *if:*

(1) ϕ_n satisfy $\int\limits_{-\infty}^{\infty} \phi_n(x)\cdot dx = 1$ (that is have total mass 1), and

(2) for any fixed $\varepsilon > 0$, the functions ϕ_n are supported on $[-\varepsilon, \varepsilon]$ for n sufficiently large.

Exercise 1.2.16. *An approximation of identity weakly converges to δ_0.*

The reason for the name "approximation of identity" is that δ_0 is the identity for the convolution operation that we will define later.
Such sequences can be generated, for example, by starting with a non-negative, continuous, compactly supported function ϕ_1 of total integral 1, and by then setting $\phi_n(x) = n\phi_1(nx)$.

Exercise 1.2.17. *Find an approximation of identity.*

Note that given $\eta \in C^{-\infty}(\mathbb{R})$ of the form $\eta = \xi_f$, we can recover the value of f at t via $\lim\limits_{n\to\infty} \langle \xi_f, \phi_n(x-t)\rangle = \lim\limits_{n\to\infty} \int\limits_{-\infty}^{\infty} f(x)\phi_n(x-t)dx = f(t)$.

1.3. **Remarks on operations on distributions.** In general, many spaces of functions can be defined as completions of C_c^∞ with respect to various topologies. From this point of view, in order to define an operation on functions such a space, it is enough to define this operation for functions in C_c^∞ and prove that it is continuous with respect to the relevant topology.

For defining operations on distributions we will often use a different approach. Suppose that we have an operation α on $C_c^\infty(\mathbb{R})$, and we would like to extend it to generalized functions. We can try to do it in the following way. Given $\phi \in C_c^\infty(\mathbb{R})$, we can try to express $\langle \xi_{\alpha(f)}, \varphi\rangle$ in terms of the pairing of ξ_f with various functions in $C_c^\infty(\mathbb{R})$. If we succeed, we can apply the same procedure to an arbitrary distribution ξ in place of ξ_f. Let us now apply this approach to the notion of derivative.

1.4. **Translations of generalized functions.** For a function $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$ denote by $R_t f$ the translated function $R_t f(x) = f(x - t)$. The motivation behind the formula, including the minus sign, is that the graph of the translated function is the translation of the graph by t.
This operation can be naturaly extended to generalized functions. For this purpose observe that $\langle \xi_{R_t f}, \phi\rangle = \langle \xi_f, R_{-t}\phi\rangle$. This motivates the following definition.

Definition 1.4.1. *Given a generalized function ξ, we define its translation $R_t\xi$ by*

$$\langle R_t\xi, \phi\rangle := \langle \xi, R_{-t}\phi\rangle$$

Exercise 1.4.2. *Define the notion of translation of a generalized function using approximation by $C_c^\infty(\mathbb{R})$. In other words, prove that if $\{f_n\}$ is a weakly Cauchy sequence in $C_c^\infty(\mathbb{R})$ then so is $\{R_t f_n\}$.*

1.5. **Derivatives of generalized functions.** Let $f, \phi \in C_c^\infty(\mathbb{R})$. Since $\langle \xi_{f'}, \phi\rangle = \int\limits_{-\infty}^{\infty} f'(x) \cdot \phi(x)dx$ we can use integration by parts to deduce that

$$\langle \xi_{f'}, \phi\rangle = f(x) \cdot \phi(x)|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} f(x) \cdot \phi'(x)dx.$$

However, since ϕ and f have compact support, we know that $f(x) \cdot \phi(x)|_{-\infty}^{\infty} = 0$. Thus, $\langle \xi_{f'}, \phi\rangle = -\langle f, \phi'\rangle$. This motivates the following definition.

Definition 1.5.1. *For any $\xi \in C^{-\infty}(\mathbb{R})$, define its* derivative *$\xi' \in C^{-\infty}(\mathbb{R})$ by*

$$\langle \xi', \phi\rangle := -\langle \xi, \phi'\rangle$$

for any $\phi \in C_c^\infty(\mathbb{R})$.

For example, the derivative of δ_0 can be intuitively described as

$$\delta_0'(x) := \begin{cases} \infty & x \to 0^- \\ -\infty & x \to 0^+ \\ 0 & otherwise. \end{cases}$$

Example 1.5.2. *We have*

$$\langle \delta_0^{(n)}, \phi\rangle = (-1)^n \langle \delta_0, \phi^{(n)}\rangle = (-1)^n \phi^{(n)}(0).$$

Exercise 1.5.3. *Find a function $F \in L^1_{\mathrm{Loc}}$ for which $F' = \delta_0$ as generalized functions.*

Exercise 1.5.4. *Define the notion of derivative of a generalized function using approximation by $C_c^\infty(\mathbb{R})$. In other words, prove that if $\{f_n\}$ is a weakly Cauchy sequence in $C_c^\infty(\mathbb{R})$ then so is $\{f_n'\}$.*

The following exercise enables to define the notion of derivative of a generalized function using translations.

Exercise 1.5.5. *Prove that $\frac{\xi - R_t\xi}{t}$ weakly converges to ξ' when $t \to 0$. By this we mean that for any test function $\phi \in C_c^\infty(\mathbb{R})$ we have $\langle \xi', \phi\rangle = \lim_{t\to 0}\langle \frac{\xi - R_t\xi}{t}, \phi\rangle$.*

1.6. **The support of generalized functions.** Let $U \subset \mathbb{R}$ be an open set and let $C_c^\infty(U)$ be the space of smooth functions $f : U \to \mathbb{R}$ supported in some compact subset of U. Given a compact subset $K \subset \mathbb{R}$, we denote by $C_K^\infty(\mathbb{R})$ the space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(f) \subseteq K$. In particular $C_K^\infty(X) \subseteq C_c^\infty(\mathbb{R})$ for every compact $K \subseteq \mathbb{R}$.
We cannot evaluate a generalized function at a point. Therefore, we cannot just define its support as we did before for a function by $\mathrm{supp}(f) := \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$. However, if for some neighborhood $U \subset \mathbb{R}$ we have for every $f \in C_c^\infty(U)$ that $\langle \xi, f\rangle = 0$, then it is natural to say that $\mathrm{supp}(\xi) \subseteq \mathbb{R} \smallsetminus U$. This leads us to the following definition:

Definition 1.6.1. *Let $\xi \in C^{-\infty}(\mathbb{R})$.*

(i) For an open subset $U \subset \mathbb{R}$ we say that ξ vanishes on *U if for any $f \in C_c^\infty(U)$ we have $\langle \xi, f\rangle = 0$.*
(ii) For $\xi \in C^{-\infty}(\mathbb{R})$ we define

$$\mathrm{supp}(\xi) = \mathbb{R} \smallsetminus \bigcup \{\text{open } U \subset \mathbb{R} \,|\, \xi \text{ vanishes on } U\}$$

(iii) Denote by $C_c^{-\infty}(\mathbb{R})$ the space of distributions with compact support.

The following (difficult) exercise shows that the definition is meaningful.

Exercise 1.6.2 (*)**.** *Let $\xi \in C^{-\infty}(\mathbb{R})$.*

*(i) Let U_1, U_2 be two open segments in $\mathbb{R}$. Show that if ξ vanishes on U_1 and U_2 then ξ vanishes on $U_1 \cup U_2$. **Hint:** Use partition of unity.*
(ii) Show that if I is a set of arbitrary cardinality and $\{U_\alpha\}_{\alpha\in I}$ is a collection of open subsets of $\mathbb{R}$ with compact closures and $\xi|_{U_\alpha} \equiv 0$ for any $\alpha \in I$ then $\xi|_{\bigcup_{\alpha\in I} U_\alpha} \equiv 0$.

We will discuss this in more details and in larger generality in ?? below.

Remark 1.6.3. *Note that* $\mathrm{supp}(\xi)$ *is always a closed set.*

Example 1.6.4. *The support of δ_0 is $\{0\}$.*

Remark 1.6.5. *While the support of δ_0' is also $\{0\}$, given some $f \in C^\infty(\mathbb{R})$ for which $f(0) = 0$ but $f'(0) \neq 0$, we get that $\langle \delta_0', f\rangle = -\langle \delta_0, f'\rangle = -f'(0) \neq 0$. In other words, having $f(0) = 0$ is not enough to get $\langle \xi, f\rangle = 0$, for a distribution ξ supported at $\{0\}$. However, if f vanishes at 0 with all its derivatives, it will imply $\langle \xi, f\rangle = 0$ for any ξ supported at $\{0\}$, as follows from Exercise 1.6.7 below.*

Exercise 1.6.6. *Let $\xi_1, \xi_2 \in C^{-\infty}(\mathbb{R})$ and $a, b \in \mathbb{R}$. Show that:*

(1) $\operatorname{supp}(a\xi_1 + b\xi_2) \subseteq \operatorname{supp}(\xi_1) \cup \operatorname{supp}(\xi_2)$.
(2) $\operatorname{supp}(\xi) - \operatorname{supp}(\xi)^\circ \subseteq \operatorname{supp}(\xi') \subseteq \operatorname{supp}(\xi)$.

Exercise 1.6.7. *Show that all the generalized functions $\xi \in C^{-\infty}(\mathbb{R})$ which are supported on $\{0\}$ are of the form $\sum_{i=0}^{n} c_i \delta^{(i)}$ for some $n \in \mathbb{N}$ and $c_i \in \mathbb{R}$.*
***Hint**: prove this in three steps.*

(i) *Show that there exists n such that ξ is bounded on the set $\{f \mid f^{(i)}(x) < 1 \forall x \in \mathbb{R}, \forall i < n\}$.*
(ii) *Show that there exists $k \in \mathbb{N}$ such that $\xi x^k = 0$, that is $\langle \xi x^k, f\rangle = \langle \xi, x^k f\rangle = 0$ for every $f \in C_c^\infty(\mathbb{R})$.*
(iii) *From $\xi x^k = 0$ deduce that $\xi = \sum_{i=0}^{k-1} c_i \delta_0^{(i)}$ for some $c_i \in \mathbb{R}$.*

1.7. Products and convolutions of generalized functions.

Definition 1.7.1. *Let $f \in C^\infty(\mathbb{R})$ and $\xi \in C^{-\infty}(\mathbb{R})$. We would like to have $(f \cdot \xi)(\phi) = \int_{-\infty}^{\infty} \xi(x) \cdot f(x) \cdot \phi(x) dx$. Thus, we define $(f \cdot \xi)(\phi) := \xi(f \cdot \phi)$.*

While we can multiply every smooth function f by any generalized function ξ, the product of two generalized functions is not always defined. Notice that indeed the product of two weakly Cauchy sequences is not always a weakly Cauchy sequence, so we might not be able to approximate the product of two generalized functions by the product of their approximations.
Recall that given two functions f, g, their *convolution* is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(t) \cdot g(x - t) dt.$$

The convolution of two smooth functions is always smooth, if it exists. In addition, if f and g have compact support, then so does $f * g$:

Exercise 1.7.2.

(1) *Show that* $\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$, *where* $\operatorname{supp}(f) + \operatorname{supp}(g)$ *is the* Minkowski sum *of* $\operatorname{supp}(f)$ *and* $\operatorname{supp}(g)$. *Thus $f, g \in C_c^\infty(\mathbb{R})$ implies $f * g \in C_c^\infty(\mathbb{R})$.*
(2) *Find an example in which the left hand side is strictly contained in the right hand side.*

Given $f, g \in C_c^\infty(\mathbb{R})$ we can write $(f * g)(x) = \langle \xi_f, \tilde{g}_x \rangle$, where $\tilde{g}_x(t) := g(x-t)$. This motivates us to define the convolution $\xi * g$ as the function $(\xi * g)(x) = \langle \xi, \tilde{g_x} \rangle$. Note that the convolution of a smooth function and a generalized function is always a smooth function:

Exercise 1.7.3. *Show that for $\phi \in C_c^\infty(\mathbb{R})$ and $\xi \in C^{-\infty}(\mathbb{R})$ we get that $\xi * \phi$ is a smooth function.*

Let us now define the convolution of two generalized functions. This will not be defined for every pair of generalized functions, but for pairs such that at least one of the generalized functions have compact support. Firstly, for $f, g \in C_c^\infty(\mathbb{R})$ we would like to have $\xi_f * \xi_g = \xi_{f*g}$. This means

$$\langle \xi_f * \xi_g, \phi \rangle = \int_{-\infty}^{\infty} (f * g)(x) \cdot \phi(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot g(x-t) \cdot \phi(x) dt dx.$$

We would like like to express the right-hand side in terms of convolutions of distributions with functions. For this purpose, for a function $h \in C^\infty(\mathbb{R})$ denote $\overline{h} := h(-x)$. In these terms we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot g(x-t) \cdot \phi(x) dt dx = \int_{-\infty}^{\infty} f(t) \int_{-\infty}^{\infty} g(x-t) \cdot \bar{\phi}(-x) dx dt = \int_{-\infty}^{\infty} f(t) \cdot (\xi_g * \bar{\phi})(-t) dt = \xi_f(\overline{\xi_g * \bar{\phi}}).$$

Definition 1.7.4. *We define $\langle \xi_f * \xi_g, \phi \rangle := \langle \xi_f, \overline{\xi_g * \bar{\phi}} \rangle$.*

However, some formal justification is required. Given a compact $K \subset \mathbb{R}$, we say ρ is a *cutoff function of* K if $\rho|_K \equiv 1$ and $\rho|_V \equiv 0$, where $\mathbb{R} \backslash V$ has compact closure.

Exercise 1.7.5. *Let K and V be as above. Show that there exists a continuous cutoff function. **Hint:** use Urysohn's Lemma.*

Thus, given some $\xi \in C_c^{-\infty}(\mathbb{R})$ with $\mathrm{supp}(\xi) \subset K$ we have that $\xi(\phi) = \xi(\rho_K \cdot \phi)$. This enables us to define ξ as a functional over all $C^\infty(\mathbb{R})$ and not only on $C_c^\infty(\mathbb{R})$. For every $\phi \in C^\infty(\mathbb{R})$ we define $\xi(\phi) = \xi(\rho_K \cdot \phi)$ where $K := supp(\xi) \subset \mathbb{R}$.

Exercise 1.7.6. *Let $\xi \in C^{-\infty}(\mathbb{R})$. In an exercise above we showed: if $\phi \in C_c^\infty(\mathbb{R})$ then the convolution $\xi * \phi$ is smooth. Show that if ϕ is smooth, and* $\mathrm{supp}(\xi)$ *is compact, then $\xi * \phi$ is still smooth.*

To summarize, the convolution of two compactly supported distributions is well defined and compactly supported, while the convolution of a compactly supported distribution with an arbitrary distribution is well defined, but usually not compactly supported.

Exercise 1.7.7. *Show the following identities for any compactly supported distributions ξ_1, ξ_2 and ξ_3 in $C^{-\infty}(\mathbb{R})$.*

(1) $\delta_0 * \xi_1 = \xi_1$.
(2) $\delta_0' * \xi_1 = \xi_1'$.
(3) $\xi_1 * \xi_2 = \xi_2 * \xi_1$.
(4) $\xi_1 * (\xi_2 * \xi_3) = (\xi_1 * \xi_2) * \xi_3$.
(5) $(\xi_1 * \xi_2)' = \xi_1 * \xi_2' = \xi_1' * \xi_2$.

Exercise 1.7.8. *Let $K \subset \mathbb{R}$ be a compact set. Construct a function $f \in C_c^\infty(\mathbb{R})$ such that $f_{|_K} \equiv 1$ and $f_{|_U} \equiv 0$ for some neighborhood $K \subset U$ (Hint: convolve a suitable approximation of identity with the indicator function of K).*

1.8. **Generalized functions on $\mathbb{R}^n$.** All the notions above make sense for functions and generalized functions in several variables. The definitions and the statements literally generalize to this case. For example, let us restate the definition of convergence in $C_c^\infty(\mathbb{R}^n)$.

Definition 1.8.1 (Convergence in $C_c^\infty(\mathbb{R}^n)$)**.** *Given $f \in C_c^\infty(\mathbb{R})$ and a sequence $\{f_n\}_{n=1}^\infty$ of smooth functions with compact support, we say that $\{f_n\}_{n=1}^\infty$ converges to f in $C_c^\infty(\mathbb{R}^n)$ if:*

(1) *There exists a compact set $K \subset \mathbb{R}^n$ for which $\bigcup_{n\in\mathbb{N}} \operatorname{supp}(f_n) \subseteq K$.*
(2) *For every multi-index α, the partial derivatives $(f_n^{(\alpha)})_{n=1}^\infty$ converge uniformly to the partial derivative $f^{(\alpha)}$.*

1.9. **Generalized functions and differential operators.** A differential equation is given by the equality $Af = g$, where A is a differential operator. Assume A is a linear differential operator which is invariant under translations, i.e. we have that $AR_t(f) = R_t(Af)$. Equivalently, we assume that A has constant coefficients, e.g. $Af := f'' + 5f' + 6f$.
Though classically one was interested in solutions in smooth functions, one can also look for solutions in generalized functions. In this case, the right-hand side can also be a generalized function. As we shall see, this approach is useful even if eventually we are interested only in smooth functions.
An important special case is the equation $Af = \delta_0$. Given a solution $f = G$, using the invariance of A under translations, we get that $AG_x = \delta_x$, for $G_x := R_x G$. Exercise 1.7.7 implies that $A(f * h) = (Af) * h$ for any two functions f, h and thus $A(G * g) = AG * g = \delta_0 * g = g$. Hence, we can find a general solution f for $Af = g$ by solving a single simpler equation $AG = \delta_0$. The solution G is called *Green's function of the operator.*

Exercise 1.9.1. *Let A be a differential operator with constant coefficients (i.e. as above).*

(1) *Choose any solution for the equation $AG = \delta_0$, and describe the conditions G has to meet without using generalized functions.*

(2) *Without using generalized functions, explain the equation $A(G * g) = g$ we got for the solution G.*
(3) *Solve the equation $\Delta f = \delta_0$ (where $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian).*

1.10. **Regularization of generalized functions.**

Definition 1.10.1. *Let $\{\xi_\lambda\}_{\lambda\in\mathbb{C}}$ be a family of generalized functions. We say that the family is* analytic *if $\langle \xi_\lambda, f\rangle$ is analytic as a function of $\lambda \in \mathbb{C}$ for every $f \in C_c^\infty(\mathbb{R})$.*

Example 1.10.2. *We denote*

$$x_+^\lambda := \begin{cases} x^\lambda & x > 0 \\ 0 & x \le 0 \end{cases},$$

and define the family by $\xi_\lambda := x_+^\lambda$ for $Re(\lambda) > -1$. The behavior of the function changes as λ changes: When $Re(\lambda) > 0$ we have a continuous function; if $Re(\lambda) = 0$ we get a step function and for $Re(\lambda) \in (-1, 0)$, the function x_+^λ will not be locally bounded. However, for any λ with $Re(\lambda) > -1$, $\xi_\lambda \in L^1_{loc}(\mathbb{R})$.
We would like to extend the definition analytically to the entire complex plane. To do that we will have to leave the world of locally L^1 functions.
Deriving x_+^λ (both as a complex function or as we defined for generalized function) we obtain $\xi'_\lambda = \lambda \cdot \xi_{\lambda-1}$. This is a functional equation which enables us to ***define*** *$\xi_\lambda := \frac{\xi'_{\lambda+1}}{\lambda+1}$, and in this way extend ξ_λ to every $\lambda \in \mathbb{C}$ with $Re(\lambda) > -2$, except $\lambda = -1$. We can use this procedure interatively to extend ξ_λ to the entire complex plane. This extension is not analytic, but it is meromorphic: it has a pole in $\lambda = -1$, and by the extension formula, in $\lambda = -2, -3, \ldots$.*

This is an example of *a meromorphic family of generalized functions*.

Definition 1.10.3. *A family $\{\xi_\lambda\}_{\lambda\in\mathbb{C}}$ of generalized functions is called* meromorphic *if there exists an entire function f on $\mathbb{C}$ such that the family $f(\lambda)\xi_\lambda$ is analytic (possibly after redefining it in a discrete set of points in $\mathbb{C}$). In fact, the family ξ_λ can be undefined in a discrete set of points, as in the case of meromorphic functions.*

Exercise 1.10.4. *Show that any meromorphic family of generalized functions can be expanded into Laurent series in a punctured neighborhood of every pole, with coefficients being generalized functions.*

Exercise 1.10.5. *Find the order and the leading coefficient of every pole of $\xi_\lambda := x_+^\lambda$.*

Example 1.10.6. *For a given $p \in \mathbb{C}[x_1, \ldots x_n]$, similarly to before set,*

$$p_+(x_1, \ldots x_n)^\lambda := \begin{cases} p(x_1, \ldots x_n)^\lambda & x > 0 \\ 0 & x \le 0 \end{cases}.$$

The problem of finding the meromorphic continuation of a general polynomial was open for some time. It was solved by J. Bernstein (see [**?**]*) by proving that there exists a differential operator* $Dp_+^\lambda := b(\lambda) \cdot p_+^{\lambda-1}$*, where* $b(\lambda)$ *is a polynomial pointing on the location of the poles.* [1]

Exercise 1.10.7. *Solve the problem of finding an analytic continuation for* $p_+(x_1, \dots x_n)^\lambda$ *in the following cases:*

(1) $p(x, y, z) := x^2 + y^2 + z^2 - a$ *and* $a \in \mathbb{R}$.
(2) $p(x, y, z) := x^2 + y^2 - z^2$.

[1] There was a slightly earlier proof in [**?**, **?**], using resolution of singularities.

2. Topological properties of $C_c^\infty(\mathbb{R}^n)$

We want to analyze the space of distributions $C^{-\infty}(\mathbb{R}^n)$. For this aim, we want to introduce a topology on this space.

2.1. Normed spaces.

Definition 2.1.1. *A* normed space *over* $\mathbb{R}$ *is a vector space* V *over* $\mathbb{R}$ *with a function* $||\cdot|| : V \to \mathbb{R}_{\geq 0}$ *satisfying*

(i) $||\lambda v|| = |\lambda| \cdot ||v||$
(ii) $||v+w|| \leq ||v|| + ||w||$
(iii) $||v|| = 0 \iff v = 0$

If we weaken (iii) to state only $||0|| = 0$ *we will get the definition of a* semi-norm.

The norm defines a Hausdorff topology on V be letting the basis for open sets be the open balls $B(x,\varepsilon) := \{v \in V \mid ||v-x|| < \varepsilon\}$ for all $x \in V$, $\varepsilon \in \mathbb{R}_{>0}$.

Example 2.1.2. *(i)* $l^p := \{$*sequences* x_n *in* $\mathbb{R} \mid \sum |x_n|^p < \infty\}$
(ii) $L^p(\mathbb{R}) := \{$*measurable* $f : \mathbb{R} \to \mathbb{R} \mid |f|^p$ *is integrable on* $\mathbb{R}\}$.
(iii) $C^k(\mathbb{R}) :=$ *functions with* k *continuous bounded derivatives,*

$$||f|| := \sum_{i=1}^{k} \sup_{x\in\mathbb{R}} |f^{(i)}(x)|.$$

There are several facts that we are used to in the Euclidean space $\mathbb{R}^n$. For example, the closed bounded sets are compact, linear subspaces are always closed, and linear operators are always continuous. These properties still hold for finite-dimensional normed spaces, but fail for infinite-dimensional ones.
Let us now discuss this in more detail. Fix a normed space V.

Exercise 2.1.3. *If* $\dim V$ *is finite then the closed unit ball* $B := \{v \in V \mid ||v|| \leq 1\}$ *is compact.*

Corollary 2.1.4. *Any finite-dimensional subspace of* V *is closed.*

Proof. Let $W \subset V$ be a finite-dimensional subspace. Let $v \in \overline{W}$, and let K be the ball in W with center at 0 and radius $2||v||$. Then v lies in the closure of K. On the other hand, K is compact and thus closed in V. Thus $v \in K \subset W$. $\square$

Exercise 2.1.5. *Show that* $C^1(\mathbb{R}) \subset C^0(\mathbb{R})$ *is dense (and thus not closed).*

Exercise 2.1.6. *Find a non-continuous linear functional on* $C^0(\mathbb{R})$. Hint: *Show, using Zorn's lemma, that any linear functional on a subspace can be extended. Then, use the embedding* $C^1(\mathbb{R}) \subset C^0(\mathbb{R})$.

Exercise 2.1.7. *Show that for any two normed spaces of the same finite dimension there exists a linear homeomorphism between them.*

Proposition 2.1.8. *If the closed unit ball $B \subset V$ is compact then* $\dim V$ *is finite.*

Proof. Note that B can be covered by open balls of radius $1/2$: $B \subset \bigcup_{x \in B} B(x, 1/2)$. If B is compact then this cover has a finite subcover. Denote the centers of the subcover by $\{x_i\}_{i=1}^n$, and let $W := Span(\{x_i\}_{i=1}^n)$. Then

$$B \subset \bigcup_{i=1}^{n} B(x_i, 1/2) \subset W + 1/2B \subset W + 1/4B \subset \dots$$

Thus, for any $k \in \mathbb{N}$, any $v \in B$ can be presented as $w_k + z_k$ where $w_k \in W$ and $||z_k|| < 2^{-k}$. Thus, $v = \lim w_k$. This shows that W is dense in V. Since W is finite-dimensional, Corollary 2.1.4 implies that it is closed. Thus $V = W$ and thus $\dim V = n$. □

2.2. **Topological vector spaces.**

Definition 2.2.1. *A* topological vector space *(or linear topological space) is a linear space with a topology such that multiplication by scalar and vectors addition are continuous. More precisely, the addition and the multiplication define continuous maps:*

(i) $+ : V \times V \to V$,
(ii) $\cdot : \mathbb{R} \times V \to V$.

Remark 2.2.2. *In this definition V is a vector space over $\mathbb{R}$, but in the same way one defines topological vector spaces over any topological field, e.g. over $\mathbb{C}$ or over the field of p-adic numbers that we will define later.*

Since addition is continuous, so is translation by a constant vector. This makes all points of a topological vector space similar - the open neighborhoods of every point x are, roughly speaking, the same as those of 0. Thus, in order to define a topology on a vector space it is enough to give a basis for the topology at zero.
We assume the topological vectors spaces we consider are Hausdorff. Note that given a non-Hausdorff space V, we can quotient V by the closure of $\{0\}$ and get a Hausdorff space.

Definition 2.2.3. *Let V be a topological vector space over $\mathbb{R}$.*

(i) We say that a set $A \subseteq V$ is convex if for every $a, b \in A$ the linear combination $ta + (1-t)b \in A$ for any $t \in [0, 1]$.
(ii) We say that V is locally convex *if it has a basis of its topology which consists of convex sets.*
(iii) We say that a set $W \subseteq V$ is balanced if $\lambda W \subseteq W$ for all $\lambda \in \mathbb{R}$ satisfying $|\lambda| \leq 1$. Note that a convex set C is balanced $\iff$ it is symmetric ($C = -C$).
(iv) For every open convex balanced set C in V and $x \in V$ we define a function

$$N_C(x) = \inf\{\alpha \in \mathbb{R}_{>0} : \frac{x}{\alpha} \in C\}.$$

Exercise 2.2.4. *Let V be a topological vector space over $\mathbb{R}$.*

(i) Show that for every neighborhood U of 0 there exists an open balanced set W such that $0 \in W \subseteq U$.
(ii) (*)Find a topological vector space which is not locally convex (not necessarily of finite dimension).
(iii) Prove that V is Hausdorff $\iff$ $\{0\}$ is a closed set.
(iv) Show that if V is finite dimensional and Hausdorff then it is isomorphic to $\mathbb{R}^n$ (as a topological vector space).

Remark 2.2.5. *Note that the previous exercise shows that a linear topological space satisfies the T_1 separation axiom $\iff$ it satisfies T_2.*

Exercise 2.2.6. *Let C be an balanced open convex set in a topological vector space V. Show that*

(i) $N_C(x) < \infty$ for all $x \in V$.
(ii) $N_C(x)$ is a semi-norm.

In a locally convex space we have a basis for the topology consisting of convex sets. By Exercise 2.2.4(i) we can assume that all the sets in this basis are balanced. Thus we obtain the next corollary.

Corollary 2.2.7 (Proof - exercise)**.** *The following are equivalent.*

(i) V is locally convex.
(ii) There exists a collection of semi-norms n_α on V such that the following sets form a basis for the topology at zero:

$$B_\alpha(r) := \{x \in V \mid n_\alpha(x) < r\}$$

(iii) There exists a collection of semi-norms n_α on V such that the topology on V is the weakest topology such that all the semi-norms n_α are continuous functions $V \to \mathbb{R}_{\geq 0}$.

Exercise 2.2.8. *Find a locally convex (Hausdorff) topological vector space V such that V has no continuous norm on it. That is, every convex open set C contains a line span$\{v\}$, so $N_C(v) = 0$.*

Note however that by Hausdorffness and Corollary 2.2.7, for every vector $v \in V$ there exists a semi-norm n on V such that $n(v) > 0$.
Generalizing the proof of Proposition 2.1.8, one can prove the following theorem.

Theorem 2.2.9 ([Rud06, Theorem 1.22])**.** *Every locally compact topological vector space has finite dimension.*

2.3. **Defining completeness.** Recall that one can define the notion of Cauchy sequence for a metric space, but not for a general topological space. However, one can define it for any topological vector space V, in the following way.

Definition 2.3.1. *A sequence* $(x_n)_{n=1}^{\infty} \subset V$ *is called a* Cauchy sequence*, if for every neighborhood* U *of* $0 \in V$ *there is* $n_0 \in \mathbb{N}$ *such that* $m, n > n_0$ *implies* $x_n - x_m \in U$.

Remark 2.3.2. *More generally, if* X *has a* uniform topology*, then we can define a notion of a Cauchy sequence. We will not give the definition of a uniform topology, but we note that any topological group possesses a uniform topology, and indeed one can define a notion of a left (resp. right) Cauchy sequence as follows:* $(x_n)_{n=1}^{\infty}$ *is a Cauchy sequence if for every neighborhood* U *of* $e \in G$ *there is an index* $n_0 \in \mathbb{N}$ *such that* $m, n > n_0$ *implies* $x_m^{-1}x_n \in U$ *(resp.* $x_n x_m^{-1} \in U$*).*

Definition 2.3.3. *Let* V *be a topological vector space.* V *is called* sequentially complete *if every Cauchy sequence in it converges.*

The notion of sequentially complete is well suited for topological vector spaces in which the topology can be defined by sequences. Such are all the first-countable (Hausdorff) topological spaces, but not all Hausdorff topological spaces, as we will see from the following discussion.

Definition 2.3.4. *A subset* Y *of a Hausdorff topological space* X *is called* sequentially closed *if for every convergent sequence* $\{y_n\}_{n=1}^{\infty}$ *in* Y*, the limit lies in* Y.

Example 2.3.5. *(i) Show that in a first-countable (Hausdorff) topological space, a set is sequentially closed if and only if it is closed.*

(ii) Give an example of a first-countable (Hausdorff) topological space, and a sequentially closed subset that is not closed. ***Hint.*** *Take* X *be the real interval* $[0, 1]$ *with the co-countable topology on* X*, and* $Y := [0, 1)$.

Let us now define the general notion of a complete topological space.

Definition 2.3.6. *Let* V *be a topological vector space.*

(i) An embedding $i : V \hookrightarrow W$ *is called a* strict embedding *if* $i : V \hookrightarrow i(V)$ *is an isomorphism of topological vector spaces.*

(ii) V *is called* complete *if for every strict embedding* $\phi : V \hookrightarrow W$*, the image* $\phi(V)$ *is closed.*

Exercise 2.3.7 (*). *(i) Show that a complete topological vector space is always sequentially complete.*

(ii) Show that a first countable topological vector space V *is complete if and only if it is sequentially complete.*

??Solution: (i) Let $\overline{V}^{sec}$ be the sequential completion of V. The map $V \to \overline{V}^{sec}$ is a strict embedding ?? check how to define the topology on it.
(ii): Suppose that V is sequentially complete. Let κ be an ordered filtered set and $\varphi : \kappa \to V$ s.t. for every neighborhood of zero $U \subset V$ there exists $\alpha \in \kappa$ s.t. the

Minkowsky difference $\varphi(\kappa_{\geq\alpha}) - \varphi(\kappa_{\geq\alpha})$ lies in U. We want to prove that any such ?? converges. This means that there exists $y \in V$ for any open neighborhood $U \ni y$ there exists $\beta \in \kappa$ with $\varphi(\kappa_{\geq\beta}) \subset U$. Indeed, choose a descending basis U_i of the topology around zero. Let $\alpha_i \in \kappa$ be such that $\varphi(\kappa_{\geq\alpha_i}) - \varphi(\kappa_{\geq\alpha_i}) \subset h(U_i)$, where $h(U_i)$ is an open neighborhood of zero such that $h(U_i) + h(U_i) \subset U_i$. On the other hand, $\varphi(\alpha_i)$ is a Cauchy sequence. Let y be its limit. Let us show that y satisfies the condition. For that purpose, let U be a neighborhood of y. Let i be such that $\varphi(\alpha_i) \in h(U)$ and $U_i \subset h(U)$. We set $\beta = \alpha_i$. It is easy to see that it satisfies the conditions above.
Now let $\iota : V \to W$ be a strict embedding. We have to show that the image is closed. Let $y \in \overline{\mathrm{Im}\iota}$. Let $\{U_\alpha \,|\alpha \in I\}$ be a basis for topology at y. Let $F(I)$ be the set of all finite subsets of I, and for any $f \in F(I)$ denote $V_f := \cap_{\alpha \in f} U_\alpha$. For any $f \in F(I)$ choose $x_f \in f \cap \mathrm{Im}\iota$. By the discussion above there exists $z \in \mathrm{Im}\iota$ such that x_f converges to z in the sense we discussed above. This implies (??) that $z = y$.

Remark 2.3.8.

(i) *Equivalently, we can define that a space V is complete if every Cauchy net is convergent. From this definition it can be easily seen that any complete space X is also sequentially complete.*

(ii) *In the category of first countable topological vector spaces, completeness is equivalent to sequentially completeness, and indeed, there the notion of a Cauchy net is equivalent to the notion of a Cauchy sequence, and a set $Y \subseteq X$ is closed $\iff$ it is sequentially complete.*

Exercise 2.3.9. *Find a sequentially complete space which is not complete.* ***Hint:*** *see the above example.*

Definition 2.3.10. *Let V be a topological vector space. A space $\bar{V}$ is a* completion *of V if $\bar{V}$ is complete and there is a strict embedding $i : V \to \bar{V}$ and $i(V)$ is dense in $\bar{V}$.*

Remark 2.3.11. *We can also use a* ***universal property*** *in order to define the completion of V. A strict (?) embedding $i : V \to \bar{V}$ is a* completion *of V if:*

(1) *$\bar{V}$ is complete.*
(2) *For every map $\psi : V \to W$ where W is complete, there is a unique map $\phi_W : \bar{V} \to W$, such that $\psi \equiv \phi_W \circ i$.*

Exercise 2.3.12 (*)**.** *Show that these two definitions of completeness are equivalent.*

It is often easier to show that a space is complete using the universal property. In this way we avoid dealing with Cauchy nets or filters. However, in order to show such completion exists one has to use these notions.

Exercise 2.3.13.

(1) ($*$) *Show that every Hausdorff topological vector space has a completion.*
(2) *Show that in the category of first countable topological vector spaces both definitions of completion are equivalent to being sequentially complete.*

2.4. **Fréchet spaces. Reminder:** A *Banach space* is a normed space, which is complete with respect to its norm. A *Hilbert space* is an inner product space which is complete with respect to its inner product.

Theorem 2.4.1. *(Hahn-Banach) Let V be a normed topological vector space, $W \subseteq V$ a linear subspace, and $C \in \mathbb{R}_{>0}$. Let $f : W \to \mathbb{R}$ be a linear functional such that $|f(x)| \leq C \cdot \|x\|$ for all $x \in W$. Then there exists $\widetilde{f} : V \to \mathbb{R}$ such that $\widetilde{f}|_W = f$ and $\left|\widetilde{f}(x)\right| \leq C \cdot \|x\|$ for all $x \in V$.*

Exercise 2.4.2. *Let V be a locally convex topological vector space (i.e, not necessarily normed), and let $f : W \to \mathbb{R}$ be a continuous linear functional, where $W \subseteq V$ is a closed linear subspace of V. Show that f can be extended to V.*

Definition 2.4.3. *The space of all continuous functionals on a topological vector space V is called the dual space and denoted by V^*.*

Exercise 2.4.4. *Let $W \subseteq V$ be infinite-dimensional vector spaces. Show that any linear functional on W can be extended to a linear functional on V. (There is no topology in this exercise).*

Definition 2.4.5. *A topological space (X, τ) is said to be metrizable if there exists a metric which induces the topology τ on X.*

Remark 2.4.6. *Every normed space is Hausdorff and locally convex, since there is a basis of its topology consisting of open balls, which are convex. We also know that every normed space is metric. However, metrizability does not force local convexity and vice versa.*

Definition: A topological space X is called a *Fréchet space* if it is a locally convex, complete space which is metrizable.

Exercise 2.4.7. *Show that for a locally convex topological vector space V the following three conditions are equivalent.*

(1) *V is metrizable.*
(2) *V is first countable (that is it has a countable basis of its topology at every point).*
(3) *There is a countable collection of semi-norms $\{n_i\}_{i\in\mathbb{N}}$ that defines the basis of the topology of V , i.e, $U_{i,\epsilon} = \{x \in V | n_i(x) < \epsilon\}$ is a basis of the topology at 0.*

Hint: given a countable family of semi-norms define a metric by

$$d(x,y) := \sum_{k=1}^{\infty} \frac{||x-y||_k}{(1+||x-y||_k)k^2}$$

Exercise 2.4.8. *Let V be a locally convex metrizable space. Prove that V is complete (and consequentially is a Fréchet space) $\iff$ it is sequentially complete.*

Recall that the completion of a normed space V with respect to its norm is the quotient space $\bar{V}$ of all Cauchy sequences in X under the equivalence relation $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty} \iff \lim_{n\to\infty} \|x_n - y_n\| = 0$. In particular, $\bar{V}$ is a Banach space. Completing V with respect to a semi-norm N results in the elimination of all elements $\{x \in V \,|\, n(x) = 0\}$. The quotient space equipped with the induced norm on the quotient then yields a Banach space.

Example 2.4.9. *Let V be the space of step functions on $\mathbb{R}$, and consider the norm $\|f\|_1 := \int_{\mathbb{R}} |f(x)|dx$. The completion of V with respect to $\|\cdot\|_1$ is isomorphic to the Banach space $L^1(\mathbb{R})$ (equipped with the norm on the quotient).*

Let V be a Fréchet space, then we have a family of semi-norms $\{n_i\}_{i\in\mathbb{N}}$ on V. We can form a new system of ascending semi-norms by replacing n_i with $\max_{j\leq i}\{n_j\}$. Let V_i be the completion of V with respect to n_i.
If n_i and n_j were norms (and not just semi-norms), which satisfy $\forall x \in V, n_i(x) \geq n_j(x)$, we would get a continuous inclusion $V_i \hookrightarrow V_j$. A sequence of ascending norms $n_1 \leq n_2 \leq \dots$ thus gives rise to a descending chain of completions $V_1 \hookleftarrow V_2 \hookleftarrow V_3 \hookleftarrow \dots$. Our space V is then an inverse limit, $V = \varprojlim V_i$, which in this case has a very nice description: it is an intersection $V = \bigcap_{i\in\mathbb{N}} V_i$ of the Banach spaces defined above.
If n_i and n_j are semi-norms, we get a continuous map $V_i \to V_j$ as every converging sequence is mapped to a converging sequence which need not be injective. In this case V will be the inverse limit $\varprojlim V_i$ where the topology on V is generated by all the sets of the form $\varphi_i^{-1}(U_i)$ where U_i is an open set in V_i and $\varphi_i : V = \varprojlim V_i \to V_i$ is the natural projection map which is part of the data of $\varprojlim V_i$.

Example 2.4.10. *The following are examples of Fréchet spaces.*

(1) $V := C^{\infty}(S^1)$ is a Fréchet space. Define the norms $\{n_i\}_{i\in\mathbb{N}}$ by $\|f\|_{n_i} := \max_{j\leq i} \sup_{x\in S^1} \{|f^{(j)}(x)|\}$. The completion with respect to n_k is $V_k = C^k(S^1)$, the space of k-times differentiable functions. This family of norms satisfies $\forall x \in V$ we have that $n_j(x) \leq n_i(x)$ if $j \leq i$, so by the argument above we indeed have $C^{\infty}(S^1) = \bigcap_{k\in\mathbb{N}} C^k(S^1)$.

(2) $V = C^{\infty}(\mathbb{R})$ is a Fréchet space. Define n_i by $\|f\|_{n_i} := \max_{j\leq i} \sup_{x\in K_i} \{|f^{(j)}(x)|\}$ where $K_i = [-i,i]$. Notice that this gives an ascending chain of seminorms

so this defines a Fréchet space $V = \varprojlim V_i$. A similar argument shows that $C^\infty(\mathbb{R}^n)$ is a Fréchet space, as well as $C^\infty(M)$ for every smooth manifold M. In these cases we take the supremum over all the possible directional derivatives.

(3) *Let K be a compact set and $n \in \mathbb{N}_0$, then $C_K^\infty(\mathbb{R}^n)$ is a Fréchet space. Let $k \in \mathbb{N}_0$, $C_K^k(\mathbb{R}^n)$ is a Banach space and in particular a Fréchet space. $C_c^\infty(\mathbb{R}^n)$ is not Fréchet.*

2.5. **Sequence spaces.** An important family of examples of Fréchet spaces are sequence spaces.

Example 2.5.1. *The space ℓ^p is the space of all sequences $(x_n)_{n=1}^\infty$ with values in $\mathbb{R}$, such that $\sum_{n=1}^\infty |x_n|^p < \infty$. It is a Banach space, and for $p = 2$ it is a Hilbert space.*
Let $\mathrm{SW}(\mathbb{N})$ be the space of all the sequences which decay to zero faster than any polynomial, i.e. $\forall n \in \mathbb{N}$, $\lim_{i\to\infty} x_i \cdot i^n = 0$. A family of norms one can consider when analyzing these spaces is $||(x_i)_{n=1}^\infty||_n = \sup_{i\in\mathbb{N}}\{|x_i \cdot i^n|\}$. It is not hard to see that with respect to these norms every Cauchy sequence converges. Define the topology on $\mathrm{SW}(\mathbb{N})$ using by the family of norms $||\cdot||_n$, then $\mathrm{SW}(\mathbb{N})$ is a Fréchet space. This is an example of a Fréchet space which is not a Banach space.

Remark 2.5.2. *How can we see every Cauchy sequence converges? Why is not it a Banach space?*
The dual space $\mathrm{SW}(\mathbb{N})^*$ is $\{(x_i)_{i=1}^\infty \mid \exists n, c : \ |x_i| < c \cdot i^n\}$. This is a union of Banach spaces, as opposed to the intersection we had when defining the completion of a Fréchet space (we will discuss the dual space more thoroughly next lecture). Note that both $\mathrm{SW}(\mathbb{N})$ and $\mathrm{SW}(\mathbb{N})^*$ contain the subspace of all sequences with compact support (that is sequences with finitely many non-zero elements).

Example 2.5.3. *Smooth functions on the unit circle, $C^\infty(S^1)$, correspond to sequences $(x_i)_{i=1}^\infty$ decaying faster than any polynomial. More precisely, we can view $f \in C^\infty(S^1)$ as a periodic function in $C^\infty(\mathbb{R})$ which can be written as $f(x) = \sum_{n=-\infty}^{\infty} a_n \cdot e^{int}$ (for functions of period 1). We thus attach to f the sequence $(a_n)_{n=1}^\infty$ where a_n decays faster then any polynomial.*

Exercise 2.5.4.

(1) *Show that the Fourier series map $\mathcal{F} : C^\infty(S^1) \to \mathrm{SW}(\mathbb{Z})$ via $f \mapsto a_n$ is an isomorphism of Fréchet spaces. In other words, show that it is a bijection and that for any semi-norm P_i of $\mathrm{SW}(\mathbb{Z})$ there exists a semi-norm S_j of $C^\infty(S^1)$ and $C \in \mathbb{R}$ such that for any $f \in C^\infty(S^1)$ we have that $\|\mathcal{F}(f)\|_{P_i} < C \cdot \|f\|_{S_j}$ (and vice-versa, or use Banach's open mapping theorem for Fréchet spaces).*

(2) *Define a Fréchet topology on*

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) | \lim_{x\to\pm\infty} f^{(n)}(x) \cdot x^k \to 0 \text{ for all } k\}.$$

2.6. **Direct limits of Fréchet spaces.**

Definition 2.6.1. *The direct limit of an ascending sequence of vector spaces is the space $V_\infty := \bigcup_{n\in\mathbb{N}} V_n$. This is not a Fréchet space, but a locally convex topological vector space. A convex subset $U \subseteq V_\infty$ is open $\iff$ $U \bigcap V_n$ is open in V_n for all n.*

Every space of the form $C^\infty(K)$ can be given the induced topology from $C^\infty(\mathbb{R})$. Taking the union of the ascending chain

$$C^\infty([-1,1]) \subset C^\infty([-2,2]) \subset \dots$$

gives all smooth functions with compact support $C_c^\infty(\mathbb{R}) = \lim_{n\to\infty} C^\infty([-n,n])$ as a direct limit. However, this is not a Fréchet space - it is a direct limit and not an inverse limit. A basis of the topology of $C_c^\infty(\mathbb{R})$ at 0 is given by the sets:

$$U_{(\epsilon_n,k_n)} := \sum_{n\in\mathbb{N}} \{f \in C^\infty(\mathbb{R}) \,|\, \mathrm{supp}(f) \subseteq [-n,n] \text{ and } |f^{(k_n)}| < \epsilon_n\},$$

where $\epsilon_n \in \mathbb{R}_{0>}$, $k_n \in \mathbb{N}_0$ and Σ denotes Minkowski sum, that is $A+B := \{a+b | a \in A, b \in B\}$.

Exercise 2.6.2. *Show that a sequence $(f_n)_{n=1}^\infty$ in $C_c^\infty(\mathbb{R})$ converges to $f \in C_c^\infty(\mathbb{R})$ with respect to the topology defined above if and only if it converges as was defined in the first lecture (Definition 1.2.2), i.e.,*

(1) *There exists a compact set $K \subseteq \mathbb{R}$ s.t. $\bigcup_{n=1}^\infty supp(f_n) \subseteq K$.*
(2) *For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$.*

Remark 2.6.3. *Notice that the topology on $C_c^\infty(\mathbb{R})$ is complicated- it is a direct limit of an inverse limit of Banach spaces!*

Exercise 2.6.4. *Show that taking a convex hull instead of a Minkowski sum (i.e., defining $U_{(\epsilon_n,k_n)} := \mathrm{conv}_{n\in\mathbb{N}}\{f \in C^\infty(\mathbb{R}) \,|\, \mathrm{supp}(f) \subseteq [-n,n], f^{(k_n)} < \epsilon_n\}$) will result in the same topology. This shows that $C_c^\infty(\mathbb{R})$ is a locally convex topological vector space (Note that this follows since this is a direct limit of Fréchet spaces).*

Finally, Fréchet spaces have several more nice properties:

- Every surjective map $\phi : V_1 \to V_2$ between Fréchet spaces is an open map (it is enough to demand that V_2 is Fréchet and V_1 is complete).
- In the previous item, defining $K := \ker\phi$, one can show that the quotient V_1/K is a Fréchet space, and ϕ factors through V_1/K, that is $\phi : V_1 \to V_1/K \to V_2$, where the map $V_1/K \to V_2$ is an isomorphism.

- Every closed map $\phi : V_1 \to V_2$ between Fréchet spaces can be similarly decomposed. Firstly, by showing $\mathrm{Im}(\phi)$ is a Fréchet space, and then by writing $V_1 \to \mathrm{Im}(\phi) \to V_2$.

2.7. Topologies on the space of distributions.

Remark 2.7.1. *Let $U \subseteq \mathbb{R}^n$ be an open set, we define $C^{-\infty}(U)$ to be the continuous dual of $C_c^\infty(U)$.*

Definition 2.7.2.

(1) Let V be a topological vector space. A subset $B \subseteq V$ is called bounded if for every open $U \subseteq V$ there exists $\lambda \in \mathbb{R}$ such that $B \subseteq \lambda \cdot U$.

(2) Denote $V^* = \{f : V \to \mathbb{R} : f \text{ is linear and continuous}\}$. There are many topologies one can define on V^*, we mention here two of those. Let $\epsilon > 0$ and $S \subseteq V$, and set $U_{\epsilon,S} = \{f \in V^* : \forall x \in S,\ |f(x)| < \epsilon\}$.

(a) The weak topology on V^*, denoted V_w^*. The basis for the topology on V_w^* at 0 is given by:

$$\mathcal{B}_w := \{U_{\epsilon,S} : \epsilon > 0,\ S \text{ is finite}\}.$$

(b) The strong topology on V^*, denoted V_S^*. The basis for the topology on V_S^* at 0 is given by:

$$\mathcal{B}_S := \{U_{\epsilon,S} : \epsilon > 0,\ S \text{ is bounded}\}.$$

In particular, every open set in V_w^* is open in V_s^*.

By definition, a sequence $\{f_n\}_{n=1}^\infty$ in V^* converges to $f \in V^*$ if and only if for every $U_{\epsilon,S} \in \mathcal{B}$ there exists $N \in \mathbb{N}$ s.t. $(f_n - f) \in U_{\epsilon,S}$ for $n > N$. That is, $\forall x \in S$ we have that $|f_n(x) - f(x)| < \epsilon$. Therefore $(f_n)_{n=1}^\infty$ converges to f w.r.t the weak topology $\iff$ it converges point-wise, and it converges to f w.r.t the strong topology $\iff$ it converges uniformly on every bounded set.

Remark 2.7.3. *If the topology on V is given by a collection of semi-norms (such as in Fréchet spaces), a set is bounded if and only if it is bounded with respect to every semi-norm.*

Theorem 2.7.4. *(Banach-Steinhaus) Let V be a Fréchet space, W be a normed vector space, and let F be a family of bounded linear operators $T_\alpha : V \to W$. If for all $v \in V$ we have that $\sup\limits_{T \in F} \|T(v)\|_W < \infty$ then there exists k such that*

$$\sup_{T \in F, \|v\|_k = 1} \|T(v)\|_W < \infty.$$

(we assume that $||v||_k \leq ||v||_{k+1} \, \forall k$).

Example 2.7.5 (Fleeing bump function)**.** *Let $V = \mathbb{R}$ and let ψ be a bump function. Notice that $g_n(x) = \psi(x + n)$ converges pointwise to 0 (and hence also in the weak* *topology). Note that g_n does not converge uniformly to 0, but it **does** converge uniformly on bounded sets to 0, so it strongly converges to 0.*

Assume V is a Fréchet space. Recall that we can define V as an inverse limit of Banach spaces $V = \bigcap_{i=1}^{\infty} V_i$ where V_i is the completion of V with respect to an increasing sequence of semi-norms n_i. If we dualize the system $\{V_i\}_{n=1}^{\infty}$ we get an increasing sequence $V_1^* \subseteq V_2^* \subseteq \ldots \subseteq V_S^* = \varinjlim V_i^*$, and get that V_S^* is a direct limit of Banach spaces (as a topological vector space).

Exercise 2.7.6. *Let $S \subseteq C_c^\infty(\mathbb{R})$ be a bounded set, then there exists a compact $K \subset \mathbb{R}$ such that $S \subseteq C_K^\infty(\mathbb{R})$.*

Exercise 2.7.7. *Consider the embedding $C_c^\infty(\mathbb{R}) \hookrightarrow C^{-\infty}(\mathbb{R})$, defined by $f \mapsto \xi_f$. Show that:*
1) This embedding is dense with respect to the weak topology on $C^{-\infty}(\mathbb{R})$.
2) This embedding is dense with respect to the strong topology on $C^{-\infty}(\mathbb{R})$.
3) $C^{-\infty}(\mathbb{R})$ with the weak topology is sequentially complete but not complete.
4) $\overline{C^{-\infty}(\mathbb{R})_w} = C_c^\infty(\mathbb{R})^\#$ where the latter is the full dual space (that is all functionals, not necessarily continuous).
5) $C^{-\infty}(\mathbb{R})_S$ is complete.

3. Geometric properties of $C^{-\infty}(\mathbb{R}^n)$

3.1. Sheaf of distributions.

Definition 3.1.1. *Let $U_1 \subseteq U_2 \subseteq \mathbb{R}^n$ be open sets. Every function $f \in C_c^\infty(U_1)$ can be extended to a function $\mathrm{ext}_0 f \in C_c^\infty(U_2)$ by defining $\mathrm{ext}_0 f\mid_{U_2\setminus U_1} \equiv 0$, hence we have an embedding $C_c^\infty(U_1) \hookrightarrow C_c^\infty(U_2)$. This embedding defines a restriction map $C^{-\infty}(U_2) \to C^{-\infty}(U_1)$, mapping $\xi \mapsto \xi\mid_{U_1}$, with $\langle \xi\mid_{U_1}, f\rangle = \langle \xi, \mathrm{ext}_0 f\rangle$.*

Remark 3.1.2.

1. *For an open $U \subset \mathbb{R}^n$, the topology we defined on $C_c^\infty(U)$ is generally not the same as the induced topology when considering it as a subspace of $C_c^\infty(\mathbb{R}^n)$.*
2. *For every compact $K \subset U$, we have $C_K^\infty(U) \subset C_c^\infty(U)$. Here the topology on $C_K^\infty(U)$ is indeed the induced topology from $C_c^\infty(U)$.*

Next we prove that with respect to the restriction operation for distributions defined above, the space of distributions is equipped with a natural structure of a sheaf.

Lemma 3.1.3 (Locally finite partition of unity)**.** *Let I be an indexing set and $U = \bigcup_{i\in I} U_i$ be a union of open sets. Then there exist functions $\lambda_i \in C_c^\infty(U)$ such that:*

(i) $\mathrm{supp}(\lambda_i) \subset U_i$
(ii) For every $x \in U$, there exists an open neighborhood U_x of x in U and a finite set S_x of indices such that $\lambda_i|_{U_x} \equiv 0$ for all $i \notin S$.

(iii) For every $x \in U$, $\sum_{i\in I} \lambda_i(x) = 1$.

Proof. Since $\mathbb{R}^n$ is paracompact, we can choose a locally finite refinement V_j of U_i, i.e. a set J, a function $\alpha : J \to I$ and an open cover $\{V_j\}_{j\in J}$ of U such that $V_j \in U_{\alpha(j)}$ for any $j \in J$ and any $x \in U$ has an open neighborhood U_x that intersects only finitely many V_j. Furthermore, we can assume that V_j are open balls $B(x_j, r_j)$. Since the closures $\overline{B(x_j, r_j)}$ are compact, there exist ϵ_j such that $\{B(x_j, r_j - \epsilon_j)\}_{j\in J}$ still cover U. For any j, let ρ_j be smooth non-negative bump functions satisfying ${\rho_j}_{|B(x_j,r_j-\epsilon_j)} \equiv 1$ and $\rho_j|_{B(x_j,r_j)^c} \equiv 0$. For any i define

$$f_i(x) = \begin{cases} \frac{\rho_i(x)}{\sum_{j\in J}\rho_j(x)} & x \in U_i \\ 0 & x \notin U_i \end{cases}.$$

Note that the sum in the denominator is finite. Now, for every i define

$$\lambda_i(x) = \sum_{j\in\alpha^{-1}(i)} f_j(x).$$

□

Theorem 3.1.4. *With respect to the restriction map defined above, distributions form a sheaf, that is given an open $U \subseteq \mathbb{R}^n$, and open cover $U = \bigcup_{i\in I} U_i$, we have:*

1. *(Identity axiom) Let $\xi \in C^{-\infty}(U)$. If for every i we have that $\xi_{|U_i} \equiv 0$, then $\xi_{|U} \equiv 0$.*
2. *(Glueability axiom) Given a collection of distributions $\{\xi_i\}_{i\in I}$, where $\xi_i \in C^{-\infty}(U_i)$, that agree on intersections, i.e. $\forall i, j \in I$ we have that $\xi_i|_{U_i\cap U_j} \equiv \xi_j|_{U_i\cap U_j}$, there exists $\xi \in C^{-\infty}(U)$ satisfying $\xi_{|U_i} \equiv \xi_i$ for any i.*

Proof. Choose a locally finite partition of unity $1 = \sum_{i\in I} \lambda_i$ corresponding to the cover U_i by Lemma 3.1.3.

(1) Given $f \in C_c^\infty(U)$ we need to show $\langle \xi, f\rangle = 0$. Let $f_i := \lambda_i f$. Then $f = \sum_{i=1}^{n} f_i$, and

$$\langle \xi, f\rangle = \langle \xi, \sum_{i=1}^{n} f_i\rangle = \sum_{i=1}^{n}\langle \xi, f_i\rangle = 0.$$

(2) Note that for any compact $K \subseteq U$ we then have that $\lambda_i|_K \equiv 0$ for all but finitely many i. Now suppose we are given $\xi_i \in C^{-\infty}(U_i)$ which agree on pairwise intersections. For any $f \in C_c^\infty(U)$ define

$$\langle \xi, f\rangle := \sum_{i\in I}\langle \xi_i, \lambda_i f\rangle.$$

Since f is supported on some compact K this sum is finite. It is clear that ξ is linear, we need to prove that it is continuous, and that $\xi|_{U_i} = \xi_i$.

Let f_n converge to f, where all functions lie in $C_c^\infty(U)$. Then also $\lambda_i \cdot f_n \to \lambda_i \cdot f$ as the multiplication $(f,g) \mapsto f \cdot g$ is continuous. Since $\bigcup\limits_{n=1}^{\infty} \mathrm{supp} f_n \subseteq K$ for some $K \subseteq U$, we have $f\lambda_i \equiv 0$ for all but finitely many indices i so we can write $\langle \xi, f\rangle = \sum_{i=1}^{k} \langle \xi_i, \lambda_i f\rangle$ and $\langle \xi, f_n\rangle = \sum_{i=1}^{k} \langle \xi_i, \lambda_i f_n\rangle$ for any n. By continuity of ξ_i we get that $\langle \xi_i, \lambda_i \cdot f_n\rangle \to \langle \xi_i, \lambda_i \cdot f\rangle$ and therefore

$$\langle \xi, f_n\rangle = \sum_i \langle \xi_i, \lambda_i \cdot f_n\rangle \to \sum_i \langle \xi_i, \lambda_i \cdot f\rangle = \langle \xi, f\rangle,$$

so ξ continuous. Now let $f \in C_c^\infty(U_j)$, then

$$\langle \xi, f\rangle = \sum_i \langle \xi_i, \lambda_i f\rangle = \sum_i \langle \xi_j, \lambda_i f\rangle = \langle \xi_j, \sum_i \lambda_i f\rangle = \langle \xi_j, f\rangle,$$

where the second equality follows from the fact that $\lambda_i f \in C_c^\infty(U_j \cap U_i)$ and $\xi_i|_{U_i \cap U_j} \equiv \xi_j|_{U_i \cap U_j}$.

A second way to prove continuity of ξ is working with the open sets in the topology of $C_c^\infty(U)$. As ξ_i are continuous, they are bounded in some convex open set $0 \in B_i$, so $|\langle \xi_i, f\rangle| < \epsilon$ for every $f \in B_i$. Notice that $\mathrm{conv}(\bigcup\limits_{i\in I} B_i)$ is open in $\bigoplus_{i\in I} C_c^\infty(U_i)$, where each B_i is an open set in $C_c^\infty(U_i)$ and hence a set in $\bigoplus_{i\in I} C_c^\infty(U_i))$ as $\mathrm{conv}(\bigcup\limits_{i\in I} B_i) \cap C_c^\infty(U_i) = B_i$. Consider the map $\varphi : \bigoplus_{i\in I} C_c^\infty(U_i) \to C_c^\infty(U)$ given by extension by zero and summation. Note that $B := \varphi(\mathrm{conv}(\bigcup\limits_{i\in I} B_i))$ is open. Now let $f \in B$. We can write $f = \sum\limits_{j_i=1}^{n} a_i f_i$ where $f_i \in B_{j_i}$ and $\sum a_i = 1$. Therefore $\xi(f) := \sum \xi_i(a_i f_i) < \sum a_i \cdot \epsilon = \epsilon$ and ξ is bounded on B.

□

3.2. **Filtration on spaces of distributions.**

Exercise 3.2.1. *Let $U \subset \mathbb{R}^n$, show that in $C_c^\infty(\mathbb{R}^n)$ we have*

$$\overline{C_c^\infty(U)} = \{f \in C_c^\infty(\mathbb{R}^n) : \forall x \notin U,\ \forall \text{ differential operator } L,\ Lf(x) = 0\}.$$

Consider $U = \mathbb{R}^n \backslash \mathbb{R}^k$. We wish to describe the space of distributions supported on $\mathbb{R}^k$, which we denote $C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)$. Notice that:

$$C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n) = \{\xi \in C^{-\infty}(\mathbb{R}^n) | \forall f \in C_c^\infty(\mathbb{R}^n \backslash \mathbb{R}^k) \text{ we have } \langle \xi, f\rangle = 0\}$$

and by continuity this is the same as:

$$\{\xi \in C^{-\infty}(\mathbb{R}^n) | \forall f \in \overline{C_c^\infty(\mathbb{R}^n \backslash \mathbb{R}^k)} \text{ we have } \langle \xi, f\rangle = 0\} = \{\xi \in C^{-\infty}(\mathbb{R}^n) |\, \xi|_V = 0\},$$

where $V = \overline{C_c^\infty(U)}$ as described in Exercise 3.2.1. Notice that we can define a natural descending filtration on V by:

$$V \subseteq V_m = \{f \in C_c^\infty(\mathbb{R}^n) | \forall i \in \mathbb{N}_0^{n-k} \text{ with } |i| \leq m \text{ we have } \frac{\partial^i f}{(\partial x)^i}|_{\mathbb{R}^k} = 0\}.$$

We immediately see that $f \in V_m$ implies $f \in V_{m-1}$, hence this is a descending chain. After dualizing, this defines an ascending filtration on $C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$ by:

$$F_m(C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)) = V_m^\perp = \{\xi \in C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n) : \xi|_{V_m} = 0\} \subseteq C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n).$$

Exercise 3.2.2. *Show the following.*

(1) $\bigcap_{m\geq 0} V_m = V = \overline{C_c^\infty(\mathbb{R}^n \backslash \mathbb{R}^k)}$.
(2) $\bigcup_{m\geq 0} F_m \neq C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$.
(3) Let $U \subseteq \mathbb{R}^n$ *be open and* $\overline{U}$ *compact. Show that for every* $\xi \in C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$ *there exists* $\xi' \in F_m$ *such that* $\xi|_U = \xi'|_U$, *thus* $\bigcup_{i=0}^\infty F_i$ *covers* $C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$ *locally.*
(4) Show that F_n *is stable under coordinate changes. More generally, let* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *be a smooth proper map that fixes* $\mathbb{R}^k$. *Show that for every* $\xi \in F_i$, $\varphi^*(\xi) \in F_i$, *where* $\langle \varphi^*(\xi), f\rangle := \langle \xi, f \circ \varphi\rangle$.

Theorem 3.2.3. *As vector spaces we have* $F_m \simeq \bigoplus_{|i|\leq m} \frac{\partial^i (C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$ *where* $i \in \mathbb{N}_0^{n-k}$ *and* $\frac{\partial^i (C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$ *is the image of* $C^{-\infty}(\mathbb{R}^k)$ *under the differential operator* $\frac{\partial^i}{\partial x^i}$ *(note that we only differentiate with respect to coordinates not lying in* $\mathbb{R}^k$*).*

Proof. We prove here the statement for $m = 0$, and return to the case where $m > 0$ in section 5. Define a map $\operatorname{res}^* : C^{-\infty}(\mathbb{R}^k) \to F_0$ by $\langle \operatorname{res}^*\xi, f\rangle = \langle \xi, f|_{\mathbb{R}^k}\rangle$ for every $\xi \in C^{-\infty}(\mathbb{R}^k)$. Notice that $\operatorname{res}^*\xi(f) = 0$ for any $f \in F_0$ by definition so it is well defined.
Furthermore, res^* is injective since if $\langle \operatorname{res}^*\xi, f\rangle = \langle \xi, f|_{\mathbb{R}^k}\rangle = 0$ for all $f \in C_c^\infty(\mathbb{R}^n)$ then $\xi = 0$ since the restriction $\operatorname{res} : C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^k)$ is surjective.
It is left to prove surjectivity. Define an extension map $\operatorname{ext}^* : F_0 \to C^{-\infty}(\mathbb{R}^k)$ by $\langle \operatorname{ext}^*\eta, f\rangle = \langle \eta, \operatorname{ext}(f)\rangle$ where $\operatorname{ext}(f)_{|\mathbb{R}^k} = f$ for every $f \in C_c^\infty(\mathbb{R}^k)$. Note that this is well defined since if we choose a different extension $\operatorname{ext}'(f)$ we get that $\langle \eta, \operatorname{ext}'(f) - \operatorname{ext}(f)\rangle = 0$ since $(\operatorname{ext}'(f) - \operatorname{ext}(f))_{|\mathbb{R}^k} \equiv 0$ and thus $\operatorname{ext}^*(\eta) = \operatorname{ext}'^*(\eta)$. Also, we have that $\operatorname{ext}^*\eta$ is a continuous functional, since we can choose the extension $\operatorname{ext}(f)$ in such a way that if $(f_n)_{n=1}^\infty$ converges to f then $(\operatorname{ext}(f_n))_{n=1}^\infty$ converges to $\operatorname{ext}(f)$. Finally, note that since $\operatorname{res}^*\operatorname{ext}^*(\eta) = \eta$, we have that ext^* is indeed surjective, and we are done. $\square$

Remark 3.2.4. *Note that if we now define* $G_m = \bigoplus_{|i|\leq m} \frac{\partial^i (C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$ *we get that* $G_m \simeq F_m/F_{m-1}$.

Exercise 3.2.5. *Show that* G_m *and* $G_{(i)} = \frac{\partial^i(C^{-\infty}(\mathbb{R}^k))}{\partial x^i}$*, where* i *is some multi-index, are not invariant under changes of coordinates, that is we might have that* $\varphi(G_m) \neq G_m$ *and* $\varphi(G_{(i)}) \neq G_{(i)}$ *for a diffeomorphism* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$*.*

3.3. **Functions and distributions on a Cartesian product.** Consider the natural map

$$\varphi : C_c^\infty(\mathbb{R}^n) \otimes C_c^\infty(\mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n \times \mathbb{R}^k), \text{ given by } \varphi(f \otimes g)(x, y) \mapsto f(x)g(y).$$

Exercise 3.3.1. *Show that this map is continuous and has a dense image.*

Let us now define a natural map

$$\Phi : C^{-\infty}(\mathbb{R}^n) \otimes C^{-\infty}(\mathbb{R}^k) \to C^{-\infty}(\mathbb{R}^n \times \mathbb{R}^k) \text{ by}$$
$$\langle \Phi(\xi \otimes \eta), F\rangle := \langle \eta, f\rangle, \text{ where } f \text{ is given by } f(y) := \langle \xi, F|_{\mathbb{R}^n \times \{y\}}\rangle.$$

Exercise 3.3.2. *Show that this map is continuous and has a dense image.*

Let us now denote by $L(C^{-\infty}(\mathbb{R}^n), C^{-\infty}(\mathbb{R}^k))$ the space of all continuous linear operators, and define a natural map

$$S : C^{-\infty}(\mathbb{R}^n \times \mathbb{R}^k) \to L(C_c^\infty(\mathbb{R}^n), C^{-\infty}(\mathbb{R}^k)) \text{ by}$$
$$\langle (S(\xi))(f), g\rangle := \langle \xi, \varphi(f \otimes g)\rangle$$

Exercise 3.3.3. *Show that the map* S

(i) is continuous and has a dense image, where $L(C_c^\infty(\mathbb{R}^n), C^{-\infty}(\mathbb{R}^k))$ *is endowed with the topology of bounded convergence.*

(ii) maps $C^\infty(\mathbb{R}^n \times \mathbb{R}^k)$ *to* $L(C_c^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^k))$ *by the formula*

$$(S(f)(g))(y) = \int_{\mathbb{R}^n} f(x, y)g(x)dx.$$

Remark 3.3.4. *(i) Note that the map* S *is similar to the matrix multiplication.*

(ii) The map S *is in fact an isomorphism. This statement is the Schwartz kernel theorem, see* [Tre67, Theorem 51.7]

(iii) There are two natural topologies one can define on a tensor product: the injective one and the projective one. If the spaces are nuclear *these two topologies coincide. We will not define these notions in the present course, but all the topological vector spaces we consider are nuclear, and thus our tensor products possess natural topology. If we complete* $C^{-\infty}(\mathbb{R}^n) \otimes C^{-\infty}(\mathbb{R}^k)$ *with respect to this topology, the map* Φ *will extend, and will become an isomorphism. The analogous statement for the map* φ *does not hold, but it will hold if we omit the compact support assumption. In other words, the extension of* φ *to the completed tensor product* $C^\infty(\mathbb{R}^n)\hat{\otimes}C^\infty(\mathbb{R}^k)$ *by the same formula is an isomorphism with* $C^\infty(\mathbb{R}^n \times \mathbb{R}^k)$*, see* [Tre67, Theorem 51.6].

4. p-ADIC NUMBERS AND ℓ-SPACES

One motivation to define the p-adic numbers comes from number theory. Assume we are given a polynomial equation $p(x) = 0$ where $p \in \mathbb{Z}[x]$. If it has an integral solution $x_0 \in \mathbb{Z}$, then surely it satisfies the equation $p(x) = 0 \mod n$ for every $n \in \mathbb{N}$. Now, consider the converse question - if we know that it has a solution modulo n for every $n \in \mathbb{N}$, does it have an integral solution in characteristic zero? In some cases, such as for quadratic forms, the answer, together with demanding that there also exists a real solution, is yes (see the Hasse principle for more on this). To know whether there exists a real solution, we can use simple methods from analysis. The question of whether an equation has a solution mod n for every $n \in \mathbb{N}$ can be simplified in two steps. Firstly, by the Chinese remainder theorem it is enough to check whether the equation has a solution mod p^n for every $n \in \mathbb{N}$. The second step is then by defining a ring $\mathbb{Z}_p$, called the ring of p-adic integers, such that if there exists a solution $x \in \mathbb{Z}_p$ it implies that there is a solution mod p^n for every $n \in \mathbb{N}$. The field $\mathbb{Q}$ of p-adic numbers is then defined to be the field of fractions of $\mathbb{Z}_p$.
A different motivation for introducing the p-adic numbers comes from a more analytic point of view. One construction of the real numbers is via completing $\mathbb{Q}$ with respect to its absolute value. An interesting question is whether this can be generalized, that is what are the possible absolute value-like functions on $\mathbb{Q}$ and their completions. It turns out that besides the standard and the trivial absolute values, every absolute value (up to equivalence) is a p-adic absolute value (this is essentially Theorem 4.1.8 below). The p-adic numbers are then obtained as the completion of $\mathbb{Q}$ with respect to such an absolute value.
In this manuscript we take the second approach, starting with defining what properties we demand from an absolute value function.

4.1. **Defining p-adic numbers.**

Definition 4.1.1. *A* topological field *is a field F, together with a topology, such that addition, multiplication and the multiplicative and additive inverses are continuous operations with respect to this topology.*

Definition 4.1.2. *Given a field F, an absolute value is a function $|\cdot| : F \to \mathbb{R}_{\geq 0}$ that satisfies:*

(1) The triangle inequality : $|x + y| \leq |x| + |y|$.
(2) $|x||y| = |xy|$.
(3) $|x| = 0 \Leftrightarrow x = 0$.

If furthermore $|x+y| \leq \max\{|x|, |y|\}$, we say that $|\cdot|$ is a non-Archimedean absolute value (and Archimedean otherwise).

For topological fields we demand the absolute value to be a continuous map. Notice that every absolute value satisfies $|1| = 1$ (as $|1| = |1| \cdot |1|$, and $|1| \neq 0$).

Example 4.1.3. *The following are absolute values:*

(1) *The trivial absolute value, defined by* $|x|_0 := \begin{cases} 0 & x = 0 \\ 1 & x \neq 0. \end{cases}$

(2) *The standard absolute value on* $\mathbb{R}$*:* $|\cdot|_\infty = \begin{cases} x & x > 0 \\ -x & x \leq 0. \end{cases}$

Definition 4.1.4. *Let p be a prime number. We define the p-adic absolute value of $x \in \mathbb{Q}$ by*

$$|x|_p = \begin{cases} p^{-n}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases}$$

where $x = p^n \frac{a}{b}$ and $a, b \in \mathbb{Z}$, are coprime to p.

Exercise 4.1.5. *Show that $|\cdot|_p$ is indeed an absolute value on $\mathbb{Q}$, and that it is non-Archimedean.*

Definition 4.1.6. *Two absolute values $|\cdot|$ and $|\cdot|'$ on F are called equivalent, and denoted $|\cdot| \sim |\cdot|'$ if they induce the same topology on F.*

Exercise 4.1.7. *Let $|\cdot|$ and $|\cdot|'$ be two absolute values on a field F. Show that the following are equivalent:*

(1) *$|\cdot|$ and $|\cdot|'$ are equivalent.*
(2) *There exists $\alpha \in \mathbb{R}_{>0}$ such that $|\cdot| = (|\cdot|')^\alpha$.*
(3) *Every sequence which is Cauchy with respect to $|\cdot|$ is Cauchy with respect to $|\cdot|'$.*

Theorem 4.1.8. *[Ostrowski's Theorem] Every absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent to either $|\cdot|_p$ for a prime p, the standard absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ induced from $\mathbb{R}$, or the trivial absolute value $|\cdot|_0$.*

Proof. Let $|\cdot|$ be an absolute value, we show it must be one of the above by cases. Assume $|\cdot|$ is non-Archimedean, i.e. $|x + y| \leq \max\{|x|, |y|\}$, and set $\mathfrak{a} = \{x \in \mathbb{Z} : |x| < 1\}$. This set is non empty as $|0| = 0$, and since we assume $|\cdot|$ is non-Archimedean it is an ideal of $\mathbb{Z}$ since,

$$|x + \ldots + x| \leq |x|, \tag{1}$$

and thus if $x \in \mathfrak{a}$, meaning that $|x| < 1$, then $xy = \underbrace{x + \ldots + x}_{y \text{ times}} \in \mathfrak{a}$. Consider a prime p. If $|p| = 1$ for every prime, we get that $|x| = 1$ for every $0 \neq x \in \mathbb{Q}$, as $|\frac{1}{p}| = |p|^{-1}$, and thus $|\cdot|$ is the discrete absolute value. Thus we can assume that there exists $p \in \mathfrak{a}$ (note that for every integer $|m| \leq 1$ by (1)), implying that

$p\mathbb{Z} \subseteq \mathfrak{a} \subsetneq \mathbb{Z}$ and consequentially $p\mathbb{Z} = \mathfrak{a}$. Now, if we put $s = -\frac{\log|p|}{\log p}$, we see that $|p| = p^{-s}$. Taking $x = p^n \frac{a}{b}$ where $a, b, n \in \mathbb{Z}$ and a and b are coprime to p we get:

$$|x| = |p^n \frac{a}{b}| = |p^n| \underbrace{|\frac{a}{b}|}_{=1} = |p|^n = p^{-ns} = |x|_p^s,$$

showing $|\cdot|$ is equivalent to $|\cdot|_p$.

Now, assume $|\cdot|$ is an Archimedean absolute value. We must have that $|n| \geq 1$ for all non-zero integers $n \in \mathbb{Z}$. Otherwise, let n be the smallest positive number such that $|n| < 1$, and for every $n < x \in \mathbb{N}$ write it in base n:

$$x = a_0 + a_1 n + a_2 n^2 + \ldots + a_r n^r, \text{ for } 0 \leq a_i \leq n-1, \ n^r \leq x. \tag{2}$$

We have $|a_i| \leq a_i \leq n$, thus

$$|x| \leq \sum_{i=0}^{r} |a_i n^i| \leq \sum_{i=0}^{r} n|n|^i = \frac{n(1-|n|^{r+1})}{1-|n|} \leq \frac{n}{1-|n|}.$$

Since $\frac{n}{1-|n|}$ is independent of r, it bounds every $x > n$, and thus we must have that $|x| \leq 1$ for every $x > n$ as otherwise $|x|^k > \frac{n}{1-|n|}$ for k large enough. We get that $|x| \leq 1$ for $x < n$ in the same way by considering $n < x^k$ for k large enough. But this means that $|x| \leq 1$ for all $x \in \mathbb{Z}$, so by the previous step it is equivalent to a p-adic absolute value, in contradiction to the fact that $|\cdot|$ is Archimedean.

We can thus assume $|n| \geq 1$ for all $n \in \mathbb{N}$. Recall the number r defined in (2) and note that $r \leq \frac{\log x}{\log n}$. We now have:

$$|x| \leq \sum_{i=0}^{r} |a_i||n|^i \leq \left(1 + \frac{\log x}{\log n}\right) n|n|^{\frac{\log x}{\log n}}.$$

Using these bounds for x^k:

$$|x|^k \leq (1 + k\frac{\log x}{\log n}) n|n|^{\frac{k \log x}{\log n}},$$

implying

$$|x|^{\frac{1}{\log x}} \leq \sum_{i=0}^{r} |a_i||n|^i \leq \sqrt[k]{(1 + k\frac{\log x}{\log n}) n}|n|^{\frac{1}{\log n}}.$$

By taking $k \to \infty$, we get $|x|^{\frac{1}{\log x}} \leq |n|^{\frac{1}{\log n}}$. But by interchanging x and n we can get that $|n|^{\frac{1}{\log n}} \leq |x|^{\frac{1}{\log x}}$ (note that if $n < x$ we can repeat this process for x and n^k for k large enough). Thus $|x|^{\frac{1}{\log x}} = |n|^{\frac{1}{\log n}} = e^{\frac{\log|x|}{\log x}}$ is constant, implying that $s = \frac{\log|x|}{\log x} = \frac{\log|n|}{\log n}$ is constant. Now, note that $|x| = x^s$ for every x and get that,

$$|x| = x^s = |x|_\infty^s,$$

finishing the proof. □

Exercise 4.1.9. *Show that given a field F and an absolute value $|\cdot|$ on it the topology it defines makes F a topological field, i.e. that addition, multiplication, and the inverse operations are continuous.*

Exercise 4.1.10. *Non-Archimedean locally compact fields such as $\mathbb{Q}_p$ have some interesting properties. Prove the following two:*

(1) *For every open ball $B(x,r) = \{y \in \mathbb{Q}_p^n : |x_i - y_i| < r\}$ of radius r with center x we have that $B(x,r) = B(x,r)$ for every $x' \in B(x,r)$.*
(2) *Any two p-adic balls $B(x,r)$ and $B(x',r')$ in $\mathbb{Q}_p^n$ are either distinct or one contains the other.*

We can now define the p-adic numbers.

Definition 4.1.11. *Let p be a prime number. We define the field of p-adic numbers $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$.*

Remark 4.1.12.

(1) *The completion is defined just as we did in the case of the Archimedean norm on $\mathbb{Q}$; by equivalence classes of Cauchy sequences. Therefore, any element $a \in \mathbb{Q}_p$ is represented by a Cauchy sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ with respect to $|\cdot|_p$.*
(2) *We get a space which is an uncountable field of characteristic 0, not algebraically closed, locally compact (every point has a compact neighborhood) and totally disconnected, i.e. every connected component is a point.*

Definition 4.1.13. *We define the p-adic integers $\mathbb{Z}_p$ to be the unit disc in $\mathbb{Q}_p$, explicitly $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$.*

Exercise 4.1.14. *Show that the p-adic absolute value extends from $\mathbb{Q}$ to $\mathbb{Q}_p$, that is show that for every Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ of elements in $\mathbb{Q}$ the limit $\lim\limits_{n\to\infty} |a_n|_p$ exists.*

Remark 4.1.15. *Notice that just like $\mathbb{R}$, this completion is not algebraically closed. Try to find an equation in $\mathbb{Q}_p$ for some p which does not have a solution $\mathbb{Q}_p$.*

4.2. Misc. -not sure what to do with them (add to an appendix about p-adic numbers?)

Theorem 4.2.1 (Taken from Koeblitz Theorem 2 page 11)**.** *Every equivalence class $a \in \mathbb{Q}_p$ for which $|a|_p \le 1$ has exactly one representative Cauchy sequence of the form $\{a_i\}_{i=1}^{\infty}$ for which:*
1) $0 \le a_i < p^i$ for $i = 1, 2, \ldots$
2) $a_i \equiv a_{i+1} (mod(p^i))$ for $i = 1, 2, \ldots$

For the proof we will use the following lemma:

Lemma 4.2.2 (Taken from Koeblitz page 12)**.** *If $x \in \mathbb{Q}$ and $\|x\|_p \leq 1$, then for any i there exists an integer $\alpha \in \mathbb{Z}$ such that $\|\alpha - x\|_p \leq p^{-i}$. The integer α can be chosen in the set $\{0, 1, 2, \ldots p^i - 1\}$.*

Proof. Let $x = a/b$ written in the form where $(gcd(a, b) = 1)$. Since $\|x\|_p \leq 1$ it follows that p does not divide b and therefore b and $p^i$are relatively prime. Then we can find $m, n \in \mathbb{Z}$ such that $bm + np^i = 1$. The intuition is that bm is close to 1 up to a small p-adic length so it is a good approximation to 1 so am is a good approximation to a/b. So we pick $\alpha = am$ and get:

$$\|\alpha - x\| = \|am - a/b\| = \|a/b\| \cdot \|bm - 1\| \leq \|bm - 1\| = \left\|np^i\right\| \leq 1/p^i$$

Note that we can add multiples of p^i to α and still have

$$\left\|\alpha - k \cdot p^i - x\right\| \leq max(1/p^i, 1/p^i) \leq 1/p^i.$$

Therefore we can assume that $\alpha \in \{0, \ldots .p^i - 1\}$. $\square$

Proof of Theorem 4.2.1. At first we prove the uniqueness: If $\{a'_i\}$ is a different sequence satisfying (1) and (2) and if there exists i_0 such that $a_{i_0} \neq a'_{i_0}$ then $a_i \neq a'_i(mod(p^{i_0}))$ for every $i > i_0$. Therefore $\|a_i - a'_i\| > 1/p^{i_0}$ so $\{a'_i\}, \{a_i\}$ are not equivalent.
Now we prove existence: Suppose we have a Cauchy sequence $\{b_i\} \in \mathbb{Q}_p$, we want to find an equivalent sequence $\{a_i\}$ with the above property. Let N_j be the number such that for every $i, i' > N_j$ we have $\|b_i - b_{i'}\| < p^{-j}$, and we can choose N_j to be strictly increasing with j, and $N_j > j$. Observe that $\|b_i\| \leq 1$ if $i > N_1$. Indeed, for all $i' > N_1$ we have that $\|b_i - b_{i'}\| < 1/p$, $\|b_i\| \leq max(\|b_{i'}\| , \|b_i - b_{i'}\|)$ and for $i' \to \infty$ we have that $\|b_{i'}\| \to \|a\|_p \leq 1$.
Now we use the lemma and get a sequence $\{a_j\}$ when $0 \leq a_j < p^j$ such that $\left\|a_j - b_{N_j}\right\| < p^{-j}$. We claim that$\{a_j\}$ is equivalent to $\{b_i\}$, and satisfies the conditions of the theorem. It indeed satisfies the conditions as:

$$\|a_{j+1} - a_j\| = \left\|a_{j+1} - b_{N_{j+1}} + b_{N_{j+1}} - b_{N_j} - (a_j - b_{N_j})\right\|$$

$$\leq max(\left\|a_{j+1} - b_{N_{j+1}}\right\| , \left\|b_{N_{j+1}} - b_{N_j}\right\| , \left\|a_j - b_{N_j}\right\|) \leq p^{-j}$$

So $a_{j+1} - a_j$ has at least p^jas a common divisor as required.
Furthermore, for any j and any $i > N_j$ we have

$$\|a_i - b_i\| = \left\|a_i - a_j + a_j - b_{N_j} - (b_i - b_{N_j})\right\| \leq max(\|a_i - a_j\| , \left\|a_j - b_{N_j}\right\| , \left\|b_i - b_{N_j}\right\|) \leq p^{-j}.$$

So $\{a_i\} \backsim \{b_i\}$. $\square$

Now, if we have some $\{a\} \in \mathbb{Q}_p$ with $\|a\| \geq 1$ then there exists some m such that $\|a \cdot p^m\| \leq 1$ and we have numbers with negative powers. Therefore we can present the p-adic numbers as:

$$\mathbb{Q}_p := \{\sum_{i=-k}^{\infty} a_i \cdot p^i, \text{ where } a_i \in \{0 \dots p^i - 1\}\}.$$

We define the *ring of integers* , denoted $\mathbb{Z}_p$ as $\mathbb{Z}_p := \{x \in \mathbb{Q}_p | \left\|x\right\|_p \leq 1\}$ or equivalently $\mathbb{Z}_p := \{\sum_{i=0}^{\infty} a_i \cdot p^i, \text{ where } a_i \in \{0 \dots p^i - 1\}\}$ or equivalently $\mathbb{Z}_p := \overline{\mathbb{Z}}_{\| \|_p}$- the closure of $\mathbb{Z}$ with respect to the p-adic norm. Notice that $\mathbb{Z}_p$ is indeed a ring and that the only invertible elements are $x \in \mathbb{Z}_p$ with $\left\|x\right\|_p = 1$.

4.3. **p-adic expansions.** We want to write the p-adic expansions of elements q in $\mathbb{Q}$. If $q \in \mathbb{N}$, that's just writing its p-base expansion. For example, $(126)_5 =$ "$\dots 002001.$" Let $x := \frac{m}{n}$ be some rational number, with $(n, m) = 1$. It is enough to describe the expansion when $p \nmid n$ (that is, when $x \in \mathbb{Z}_p \cap \mathbb{Q}$) as otherwise we can multiply x by p^k for some k, calculate the expansion, and move the point k places to the left.
We can't take remainder of x modulo p, as with integers. Instead, we can calculate the fraction $x = \frac{m}{n}$ in $\mathbb{F}_{p^k}$ for $k \in \mathbb{N}$. Thus, the expansion of x in $\mathbb{Q}_p$ is calculated inductively:

- Write the digit $x_0 := [\frac{m}{n}] \in \mathbb{F}_p$.
- The nominator of the difference $\frac{m}{n} - x_0 = \frac{m - n \cdot x_0}{n}$ is divisible by p. Redefine our fraction to be $x := \frac{1}{p} \cdot (\frac{m}{n} - x_0)$, and continue inductively.

Example 4.3.1. *Calculate $\frac{1}{2} \in \mathbb{Q}_7$. We start by solving the equation $2x_0 = 1(mod 7)$. The answer is $x_0 = 4$. In the second step we calculate $\frac{1}{7}(\frac{1}{2} - 4) = x_1$. So $2 \cdot (7x_1 + 4) = 1(mod 49)$. Therefore $x_1 = 3$. We continue by induction and get the required expansion.*

Every ball in $\mathbb{Q}_p$ is a disjoint union of p balls. For $p = 2$, the ball $\mathbb{Z}_2 = B_c(0, 1) = B_o(0, 2)$ consists of numbers with no digits to the right of the point. It's a disjoint union of two balls, B_0 and B_1 - where each B_i consists of all numbers ending with the digit $'i'$. Similarly, $B_0 = B_{00} \bigcup B_{01}$, $B_1 = B_{10} \bigcup B_{11}$, where the elements in B_{ij} end with the digits $'ij'$. And so on.
This recursive structure implies p-adic integers are homeomorphic to the Cantor set.
== Appendix material ends here ==

Exercise 4.3.2. *Show that $\sum\limits_{n=0}^{\infty} a_n$ converges in $\mathbb{Q}_p \iff |a_n|_p \to 0$.*

Exercise 4.3.3. *Show $\mathbb{Z}_p$ is homeomorphic to the Cantor set as topological spaces, where the Cantor set has its usual topology induced from the real numbers. In particular this shows $\mathbb{Z}_p$ is a compact set.*

4.4. **Inverse limits.**

Definition 4.4.1. *Let* $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \dots$ *be a sequence of Abelian groups* $\{A_i\}$ *together with a system of homomorphisms* $\{f_{ij} : A_j \to A_i \mid j > i\}$*, such that* $f_{ik} = f_{ij} \circ f_{jk}$*,* $\forall i \leq j \leq k$*. An inverse limit of a sequence of Abelian groups is defined by the collection of compatible sequences:*

$$\underleftarrow{lim} A_i = \{a \in \prod_{i \in \mathbb{N}} A_i : f_{ij}(a_j) = a_i, \forall i \leq j \in \mathbb{N}\}.$$

Exercise 4.4.2. *Prove the following:*

1. *(1) Let* $A_i := \mathbb{Z}/p^i\mathbb{Z}$*, and let* f_{ij} *be the projection* $\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$*. Show that* $\varprojlim \mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}_p$ *as a topological ring.*
2. *(2)* $\mathbb{Q}_p$ *is the localization of* $\mathbb{Z}_p$ *by* p*.*
3. *(3) Prove that* $\mathbb{Q}_p$ *is homeomorphic to the Cantor set minus a point.*
4. *(4) Prove that* $\mathbb{Q}_p^n$ *and* $\mathbb{Q}_p$ *are homeomorphic.*
5. *(5) Let* $U \subset \mathbb{Q}_p^n$ *be an open set. Show that either* U *is homeomorphic to the Cantor set, or to Cantor set minus a point.*

4.5. **Haar measure and local fields.** Let X be a topological space and let $C_c(X)$ be the space of continuous functions of compact support on X. Recall that the space of continuous linear functionals $C_c(X)^*$ can be identified with the space of regular Borel measures on X.

Theorem 4.5.1 (Haar's theorem)**.** *Let* G *be a locally compact topological group. Then:*

1. *(1) There exists a measure* μ *on* G *with values in* $\mathbb{C}$ *such that* $\mu(U) = \mu(gU)$ *for any measurable set. Equivalently, there exists* $\phi \in C_c(G)^*$ *such that for any* $g \in G$ *we have that* $\phi(f) = \phi(f_g)$ *where* $f_g(x) = f(g^{-1} \cdot x)$*.*
2. *(2) This measure is unique up to a scalar.*

Exercise 4.5.2.

1. *(1) Prove Haar's theorem for* $(\mathbb{Q}_p, +)$*.*
2. *(2) Given a Haar measure* μ*, we can define another invariant measure* $\mu_a(B) = \mu(aB)$ *for any* $a \in \mathbb{Q}_p$*. Show that* $\mu_a = |a| \cdot \mu$*.*

Definition 4.5.3. *A* local field *is a non-discrete topological field which is locally compact.*

Theorem 4.5.4. *Any local field* F *is isomorphic as a topological field to one of the following :*

1. *(1)* $\mathbb{R}$ *or* $\mathbb{C}$ *(if* F *is Archimedean).*
2. *(2) A finite extension of* $\mathbb{Q}_p$ *for some prime* p *(if* F *is non-Archimedean of characteristic* 0*).*
3. *(3) The field of formal Laurent series* $\mathbb{F}_{p^s}((t)) = \{\sum_{i=-k}^{\infty} a_i t^i \mid a_i \in \mathbb{F}_{p^s}\}$ *for some prime* p *and natural number* s *(if* F *is non-Archimedean of characteristic* p*).*

Proof. The main steps of the proof are as follows:

1. Using Haar's theorem define a measure on $(F, +)$. Using this measure we define an absolute value, up to scalar multiplication, that is, we set $|a| = \alpha(a)$ where $\mu_a = \alpha(a)\mu$.
2. We show that for every local field the absolute value which was defined in (1) defines its topology.
3. We prove that every compact metric space is complete.
4. Every local field of characteristic 0 contains $\mathbb{Q}$ and one of its completions. This means that F contains $\mathbb{R}$ if it is archimedean, and $\mathbb{Q}_p$ if it is non-archimedean.
5. We show that if F has characteristic 0, then F must be a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$. Otherwise, if it is a non algebraic extension it must be non-locally-compact.
6. We show that any finite extension of $\mathbb{Q}_p$, $\mathbb{R}$ or $\mathbb{F}_q((t))$ is indeed a local field.
7. For $char(F) \neq 0$ we show that F contains a transcendental element, name it t, and show that it contains $\mathbb{F}_q((t))$. We show that F must be a finite extension of $\mathbb{F}_q((t))$.

□

4.6. Some basic properties of ℓ-spaces.

Definition 4.6.1. *An **ℓ-space** X is a Hausdorff, locally compact and totally disconnected topological space.*

Remark 4.6.2. *We usually add the demand X is σ-compact, that is it is the union of countably many compact spaces. Such a space is also sometimes called countable at ∞.*

Exercise 4.6.3. *Find a compact ℓ-space X and $U \subseteq X$ such that U is not countable at ∞.*

Exercise 4.6.4. *Show the following:*

1. *Any non-archimedean local field is an ℓ-space.*
2. *Finite products, and open or closed subsets of an ℓ-space are ℓ-spaces. Note that any subset of a totally disconnected topological space is totally disconnected.*

Proposition 4.6.5. *Let X be an ℓ-space, then it has a basis of clopen (that is closed and open) sets (i.e. it is zero-dimensional).*

Proof. Taken from [AT08, 3.1.7]. Assume we have a point $x \in \mathcal{W} \subseteq K$, with $\mathcal{W}$ open and $K = \overline{\mathcal{W}}$ compact and set $\mathcal{P}_x = \{U \subseteq K : U \text{ is clopen in } K \text{ and } x \in U\}$ and $P = \bigcap_{V \in \mathcal{P}_x} V$. Note that $K \in \mathcal{P}_x$, thus $P \neq \varnothing$.

Now, we claim that for every closed subset F of K such that $F \cap P = \varnothing$ there exists some $W \in \mathcal{P}_x$ such that $W \cap F = \varnothing$. Indeed, set $\eta = \{U \cap F : U \in \mathcal{P}_x\}$. By assumption, it is a family of non-empty closed subsets of F, and since F is compact if $\bigcap_{V \in \eta} V = \varnothing$, then there is a finite collection of V_i such that $\bigcap_{i=0}^{n} V_i = \bigcap_{i=0}^{n} U_i \cap F = \varnothing$ (note that this is an equivalent characterization of compactness via closed sets). Now set $W := \bigcap_{i=0}^{n} U_i \in \mathcal{P}_x$. Since $\mathcal{P}_x$ is closed under finite intersections, $W \in \mathcal{P}_x$.
Now we wish to show that $P = \{x\}$. Assume the contrary, i.e. $P \neq \{x\}$. P is disconnected since X is totally disconnected, so there exists non-empty closed $x \in A$ and B such that $A \cup B = P$ and $A \cap B = \varnothing$ which are open in K. Since K is regular, (Hausdorff + locally compact implies regular), there exist open disjoint sets $U \ni U$ and $V \supseteq B$ in K, where we have $F = K \backslash (U \cup V)$ closed in K and $P \cap F = \varnothing$. We showed that for such F we can find $W \in \mathcal{P}_x$ such that $F \cap W = \varnothing$. Now, observe that the open set $G = U \cap W$ is also closed in K as,

$$\overline{G} = \overline{U} \cap W \subseteq (K \backslash V) \cap (K \backslash F) = K \backslash (V \cup F) \subseteq U.$$

Therefore $\overline{G} \subseteq U \cap W = G$ (W was closed). Since $x \in G$, we have $G \in \mathcal{P}_x$, but as $G \cap B = \varnothing$, we get that $P = A \cup B$ is not contained in G, which is a contradiction, implying $P = \{x\}$.
Since for every open set $x \in O$ in K the set $K \backslash O$ is compact and $x \notin K \backslash O$, it follows from the above claim that O contains some $V \in \mathcal{P}_x$.
Now, given an open set $x \in \mathcal{A}$ in X, we have that $\mathcal{W} \cap \mathcal{A} \subset K \cap \mathcal{A}$, is open in K, and thus contains a clopen $\mathcal{U}$ in the topology of $K \cap \mathcal{A}$ from the above. Now, $\mathcal{U}$ is closed in K and thus closed in X, and open in $\overline{\mathcal{W} \cap \mathcal{A}}$ but contained in $\mathcal{W} \cap \mathcal{A}$ and thus open in X. This finishes the proof. $\square$

Exercise 4.6.6. *Show that every σ-compact, first countable ℓ-space X is homeomorphic to one of the following:*

(1) *Countable (or finite) discrete space.*
(2) *Cantor set.*
(3) *Cantor set minus a point.*
(4) *Disjoint union of* (2) *or* (3) *with* (1).

Definition 4.6.7. *A refinement of an open cover $\bigcup_{i \in I} U_i = X$ is an open cover $\{V_j\}_{j \in J}$ such that for any j, we have that $V_j \subseteq U_i$ for some i.*

Exercise 4.6.8.

(1) *Let $C \subseteq X$ be a compact subset of an ℓ-space. Then any open cover has an open compact disjoint refinement.*
(2) *Let X be a σ-compact ℓ-space, then any open cover has an open compact disjoint refinement.*

4.7. **Distributions on ℓ-spaces.**

Definition 4.7.1. *Let X be an ℓ-space. A function $f : X \to \mathbb{C}$ is said to be* smooth *if it is locally constant, that is for every point $x \in X$ there is an open neighborhood $x \in U \subseteq X$ such that the restriction $f|_U$ is constant. Similarly to the archimedean case, the space of smooth functions on X is denoted by $C^\infty(X)$.*

Proposition 4.7.2. *Let X be an ℓ-space. Show that smooth functions separate the points in X. Assuming this, the Stone-Weierstrass theorem implies that $C^\infty(X)$ is dense in the space of all continuous functions $C(X)$.*

Proof. Let $x, y \in X$. As X is Hausdorff and has a basis of open compact, sets there exists disjoint U_x and U_y which are compact and open. Set $f_{|U_x} = 1$ and $f_{|X \setminus U_x} = 0$. Then f is smooth and $f(x) = 1$, and $f(y) = 0$. $\square$

Definition 4.7.3. *The space of smooth functions with compact support, $C_c^\infty(X) \subset C^\infty(X)$, are called* Schwartz functions. *We denote them by $S(X)$. We also denote* $\mathrm{Dist}(X) = C_c^\infty(X)^* = S^*(X)$. *We consider $S(X)$ as a vector space, without any topology.*

Exercise 4.7.4. *Let X be an ℓ-space, show that $S^*(X)$ is a sheaf.*

Remark 4.7.5. *In $\mathbb{R}^n$, the Schwartz functions are the functions whose derivatives decrease faster than any polynomial, and there is a strict containment $C_c^\infty(\mathbb{R}^n) \subset S(X) \subset C^\infty(\mathbb{R}^n)$. We will define them in the next lectures.*

4.8. **Distributions supported on a subspace.** Recall that over $\mathbb{R}$, the description of distributions on a space X that are supported on a closed subspace Z is complicated (we did that using filtrations). Distributions on ℓ-spaces behave much better.

Definition 4.8.1. *Let X be an ℓ-space, we define the support of a distribution $\xi \in S^*(X)$ as we did for distributions on real spaces, by* $\mathrm{supp}(\xi) = \bigcap_{\xi_{|D_\beta^c} \equiv 0} D_\beta$, *where $D_\beta \subset X$ are taken to be closed.*

Proposition 4.8.2. *(Exact sequence of an open subset). Let $U \subseteq X$ be open and set $Z = X \setminus U$. Then $0 \to S(U) \to S(X) \to S(Z) \to 0$ is exact.*

Proof. It is clear that extension by zero $S(U) \to S(X)$ is injective, we show that $S(X) \to S(Z)$ is onto. Let $f \in S(Z)$. As f is locally constant and compactly supported, we may assume that Z is compact and has a covering by a finite number of open sets U_i (open in Z) with $f|_{U_i} = c_i$. Notice that each U_i is of the form $U_i = W_i \cap Z$, where W_i is open in X. Therefore, $Z \subseteq \bigcup_{i=1}^{n} W_i$, and as Z is compact, we may refine $\{W_i\}_{i=1}^n$ and get that $Z \subseteq \bigcup_{j=1}^m V_j$ where V_j are open, compact

mutually disjoint and $V_j \cap Z \subseteq W_i \cap Z = U_i$ for some i. We can thus extend f by setting $f(x) = c_i$ for $x \in V_j \subseteq W_i$ and zero otherwise.
It is left to prove exactness at $S(X)$. Let $f \in S(X)$ such that $f|_Z = 0$. As f is locally constant, there is an open set $Z \subseteq V$ such that $f|_V = 0$. This implies that f is supported on $Z^c = U$ and therefore $f|_U \in S(U)$. □

Corollary 4.8.3. *Let X be an l-space, and $Z \subset X$ a closed subspace. Then:*

1. *The inclusion $i : S^*(Z) \to S^*_Z(X)$ is an isomorphism.*
2. *There is an exact sequence $0 \to S^*(Z) \to S^*(X) \to S^*(X\backslash Z) \to 0$.*

Remark 4.8.4.

1. *Note that if we replace X by $\mathbb{R}$, then the map i is not onto. For example, for $Z := \{0\} \subset \mathbb{R}$, the derivatives $\delta_0^{(n)} \in S^*_Z(\mathbb{R}^n)$ but they are not in the image of i as in that case $S^*(Z) \simeq \mathbb{C}$.*
2. *This can be corrected by replacing $S^*(X)$ by $S^*_Z(X)$. Thus the following is an exact sequence:*

$$0 \to S^*_Z(X) \to S^*(X) \to S^*(X\backslash Z).$$

Exercise 4.8.5. *Let V be a vector space (not necessarily finite dimensional) over a field K, and $L \subset V$ a linear subspace. Show that $\forall f \in L^*$ $\exists g \in V^*$ such that $g|_L \equiv f$. Use Zorn's lemma.*

Proposition 4.8.6. *Let X, Y be ℓ-spaces. Given $f_1 \in S(X)$ and $f_2 \in S(Y)$, consider the bilinear map $\phi : S(X) \otimes S(Y) \to S(X \times Y)$ via*

$$(\phi(f \otimes g))(x, y) := f(x) \cdot g(y).$$

Then ϕ is an isomorphism of vector spaces.

Proof. It is easy to see that the image lies in the space of locally constant functions. We first prove this map is surjective. Let $f \in S(X \times Y)$, then $f = \sum c_{U_i \times V_i}$ and by refining $\{U_i \times V_i\}_{i=1}^n$ we may assume that they are disjoint (note we are using the fact that supp$f$ is compact). Since each term $c_{U_i \times V_i} \in \mathrm{Im}\phi$ we are done.
To show ϕ is injective, assume that

$$\phi(\sum_{i=1}^k f_i \otimes g_i))(x, y) := \sum_{i=1}^k f_i(x) \cdot g_i(y) = 0.$$

We can assume that $\{f_i\}$ are linearly independent and that $\{g_i\}$ are non zero and that k is minimal with respect to these demands. If we take some y such that $g_1(y) \neq 0$ we get that for any $x \in X$, we have $\sum_{i=1}^k f_i(x) \cdot g_i(y) = 0$. This implies that f_i are linearly dependent. Contradiction. Hence $g_i \equiv 0$ for all i, implying $f_i \otimes g_i \equiv 0$, contradicting the assumption that k is minimal. □

Define a natural map

$$S : \mathcal{S}^*(X \times Y) \to \mathrm{Hom}_F(\mathcal{S}(X), \mathcal{S}^*(Y)) \text{ by}$$

$$\langle S(\xi)(f), g\rangle := \langle \xi, \phi(f \otimes g)\rangle.$$

Exercise 4.8.7.

(i) Show that the map S is a linear isomorphism.

(ii) Assume $X = F^n, Y = F^m$ and let μ and ν be Haar measures on X and Y. Embed $C^\infty(X \times Y) \hookrightarrow \mathcal{S}^*(X \times Y)$ and $C^\infty(Y) \hookrightarrow \mathcal{S}^*(Y)$ by multiplication by the corresponding Haar measures. Then S maps $C^\infty(X \times Y)$ to $\mathrm{Hom}_F(\mathcal{S}(X), C^\infty(Y))$ by the formula

$$(S(f)(g))(y) = \int_X f(x, y)g(x)\mu.$$

Exercise 4.8.8. *Consider the natural map*

$$\Phi : \mathcal{S}^*(X) \otimes \mathcal{S}^*(Y) \to \mathcal{S}^*(X \times Y) \text{ given by}$$

$$\langle \Phi(\xi \otimes \eta), F\rangle := \langle \eta, f\rangle, \text{ where } f \text{ is given by } f(y) := \langle \xi, F|_{X\times\{y\}}\rangle.$$

(i) Show that Φ is not onto. Hint: take $X = Y = \mathbb{Z}$.

(ii) Endow $\mathcal{S}(X \times Y)$ with the weak topology, i.e. $\xi_n \to \xi$ iff $\langle \xi_n, f\rangle \to \langle \xi, f\rangle$ for every $f \in \mathcal{S}(X \times Y)$. Show that the map Φ has dense image.

5. Vector valued distributions

Definition 5.0.1. *Let F be either $\mathbb{R}$ or $\mathbb{C}$, let X be a locally compact space and let V be a vector space over F. We define $C_c^\infty(X, V)$ to be the space of smooth functions on X with compact support with values in V. Here the smoothness of a function is the usual coordinate-wise one.*

Exercise 5.0.2. *Let V be a topological vector space over F. Prove that $C_c^\infty(X, V) \cong C_c^\infty(X) \otimes_F V$ as topological vector spaces, where the topology on $C_c^\infty(X) \otimes_F V$ is given by choosing a basis to identify V with F^n and by then taking the product topology on $C_c^\infty(X) \otimes_F F^n \cong (C_c^\infty(X))^n$. In particular, this topology is independent of a choice of a basis.*

5.1. **Smooth measures.**

Recall that a measure is a σ additive map from the σ-algebra of Borel subsets of X into $\mathbb{R}$. For us, the following characterization is better:

Definition 5.1.1. *Let X be a locally compact topological space. The space of signed measures on X is $C_c(X)^*$, i.e. all continuous functionals on $C_c(X)$ (and all linear functionals if X is an ℓ-space). A signed measure is a measure if it is non-negative on non-negative functions.*

As the space $C_c(X)$ is larger than $C_c^\infty(X)$, its dual is smaller. Explicitly, $C_c(X)^* \subseteq C_c^\infty(X)^*$ where the inclusion is the dual of the dense embedding $C_c^\infty(X) \hookrightarrow C_c(X)$. If X is a group then in $C_c(X)^*$ there is a one-dimensional space of Haar measures, which for $X = \mathbb{R}^n$ is just the space of multiples of the Lebesgue measure.

Remark 5.1.2. *We usually consider the space of complex valued measures. As in the real case, it can naturally be identified with* $C_c(X)^*$.

Definition 5.1.3. *Let* V *be a locally compact vector space (note that it must be finite dimensional as otherwise it is not locally compact). The space of* Haar measures on V, *denoted* $\mathrm{Haar}(V) \subseteq C_c(V)^*$, *is the space of translation invariant measures (which exists by Haar's theorem).*

The fact that this space is one dimensional is non-trivial, but the intuition is as follows: A Borel measure on V is determined by its value on cubes with rational coordinates, as they form basis of the topology. It is not hard to see that if the measure is translation invariant, the measures of these cubes are determined by the measure of the unit cube.

Definition 5.1.4. *Let* V *be a topological vector space. A measure* μ *on* V *is called a smooth measure if* $\mu \in C^\infty(V, \mathrm{Haar}(V))$, *i.e.* $\mu = f(x)h$ *where* f *is smooth and* h *is a Haar measure. We denote this space by* $\mu^\infty(V)$, *and the space of all compactly supported measures inside it by* $\mu_c^\infty(V)$.

Exercise 5.1.5 (*). *Let* V *be a vector space over a local field and let* $\xi \in C_c^\infty(V)^*$ *be translation invariant. Prove that* ξ *is a Haar measure, i.e. show it is a measure (note that* $C_c^\infty(V)^* \supsetneq C_c(V)^*$ *so a-priori there might be translation invariant distributions which are not measures).*

Remark 5.1.6. *Note that by definition* $\mu_c^\infty(V) \simeq C_c^\infty(V) \otimes \mathrm{Haar}(V)$ *canonically. We also have that* $\mu_c^\infty(V) \simeq C_c^\infty(V)$ *by choosing a Haar measure. This isomorphism is not canonical.*

5.2. **Generalized functions versus distributions.** We are now in a position to understand the difference between generalized functions and distributions.
A *distribution* on V is a continuous functional on the space of smooth functions with compact support:

$$\mathrm{Dist}(V) := C_c^\infty(V)^*.$$

A *generalized function* is a continuous functional on the space of smooth measures with compact support on V, i.e.

$$C^{-\infty}(V) := C_c^\infty(V, \mathrm{Haar}(V))^*.$$

As functions can be integrated against smooth measures of compact support, we have a bilinear pairing

$$C_c^\infty(V, \mathrm{Haar}(V)) \times C_c^\infty(V) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}.$$

Thus we have the following picture:

$$\begin{array}{ccc} C_c^{-\infty}(V) & \overset{\simeq}{\Longleftrightarrow} & \mathrm{Dist}(V) \\ j\uparrow & & i\uparrow \\ C_c^{\infty}(V) & \overset{\simeq}{\Longleftrightarrow} & \mu_c^{\infty}(V) \end{array}$$

where the diagonals are dual to each other. Both inclusions i and j are obtained via the pairing $\langle\cdot,\cdot\rangle$.

Exercise 5.2.1. *Show that* $\mathrm{Haar}(V) \simeq \mathrm{Dist}(V)^V$ *or equivalently that* $\mathrm{Dist}(V)^V$ *is one dimensional, for any finite dimensional vector space V over a local field F.*

Definition 5.2.2. *We can also define generalized functions with values in a vector space by either:*
1) $C^{-\infty}(V,E) := C^{-\infty}(V)\otimes E$
2) $C^{-\infty}(V,E) := C_c^{\infty}(V,\mathrm{Haar}(V)\otimes E^*)^*$
and then $C^{-\infty}(V,\mathrm{Haar}(V)) := C^{-\infty}(V)\otimes\mathrm{Haar}(V) = C_c^{\infty}(V)^* = \mathrm{Dist}(V)$.

Exercise 5.2.3.

1. *Show that the two definitions of $C^{-\infty}(V,E)$ are equivalent.*
2. *Describe an embedding* $C_c^{\infty}(V,E)\hookrightarrow C^{-\infty}(V,E)$.

5.3. **Some linear algebra.**

Definition 5.3.1. *Let V be a finite dimensional vector space over a local field F.*

1. *We define the exterior algebra as*

$$\Lambda(V) = \bigoplus_{k=0}^{\infty}\Lambda^k(V),$$

where $\Lambda^0(V) = F$, and for $k>0$ we have $\Lambda^k(V) = (\bigotimes_{j=1}^{k} V)/J_k$ where J_k is the vector space generated in $\bigotimes_{j=1}^{k} V$ by the set

$$\{v_1\otimes\ldots\otimes v_k : v_i = v_j \text{ for some } i\neq j\}.$$

2. *We define the symmetric algebra $\mathrm{Sym}(V)$ as*

$$\mathrm{Sym}(V) = \bigoplus_{k=0}^{\infty}\mathrm{Sym}^k(V),$$

where $\mathrm{Sym}^0(V) = F$, and for $k>0$ we have $\mathrm{Sym}^k(V) = (\bigotimes_{j=1}^{k} V)/I_k$ where I_k is the vector space generated in $\bigotimes_{j=1}^{k} V$ by the set

$$\{v_1\otimes\ldots\otimes v_k - v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(k)} : \sigma\in S_k\}.$$

Note that this implies that the elements of the exterior algebra are anti-symmetric (i.e. $v \otimes u = -u \otimes v$), and that $\Lambda^k(V) = 0$ if $k > \dim V$, since after choosing a basis for V and decomposing an element in $\Lambda^k(V)$ to basic tensors, there must be a basis element which appears at least twice.

Definition 5.3.2. *Let V be an n-dimensional vector space over a local field F with absolute value $|\cdot|$.*

(1) We define the space of k-forms $\Omega^k(V) = \Lambda^k(V^*)$.

(2) For a 1-dimensional space V we define a real vector space

$$|V| := \{f : V^* \to \mathbb{R} : \forall \alpha \in F, f(\alpha v) = |\alpha| f(v)\}.$$

(3) We define the densities of V as

$$\mathrm{Dens}(V) := \{f : V^n \to \mathbb{R} : f(Av_1, \dots, Av_n) = |\det(A)| f(v_1, \dots, v_n)\}.$$

Now, let $\Omega^{\mathrm{top}}(V)$ be the space of anti-symmetric n-forms on V. It is a one-dimensional space, and $\Omega^{\mathrm{top}}(V) = \Lambda^n(V^*)$.

Exercise 5.3.3. *Let $\mathcal{B}$ be the space of bases of V.*

(1) Show that $\Omega^{\mathrm{top}}(V) = \{f : \mathcal{B} \to F : f(B_1) = det(M_{B_1}^{B_2}) f(B_2) \forall B_1, B_2 \in \mathcal{B}\}$ where $M_{B_1}^{B_2}$ is the respective base changing matrix.

(2) Show that $\Omega^{\mathrm{top}}(V) = \{f : V^n \to F : f(Av_1, \dots, Av_n) = det(A) f(v_1, \dots, v_n)\}$.

Definition 5.3.4. *For a finite dimensional real vector space V define the orientation line*

$$\mathrm{Ori}(V) := \{f : \mathcal{B} \to \mathbb{R} : f(B_1) = sign(det(M_{B_1}^{B_2})) \cdot f(B_2)\}.$$

Exercise 5.3.5. *Using the tensor product of the natural maps $\Omega^{\mathrm{top}}(V) \to \mathrm{Dens}(V)$ and $\Omega^n(V) \to \mathrm{Ori}(V)$ show that $\Omega^{\mathrm{top}}(V) = \mathrm{Dens}(V) \otimes \mathrm{Ori}(V)$.*

Note that the orientation line is a linear space and not just two points as one is used to think about orientations. However, we have two distinguished points in $\mathrm{Ori}(V)$, the two functions with absolute value 1. These are the usual orientations we are used to thinking about.

Proposition 5.3.6. *Show that there is a canonical isomorphism $\mathrm{Dens}(V) \simeq \mathrm{Haar}(V)$.*

Proof. A Haar measure can be viewed both as a functional on compactly supported, continuous functions and as a function on a Borel algebra. The absolute value of the determinant $|\det| : V^n \to \mathbb{R}$ is an element of the one dimensional space $|\Omega^n(V)|$ (recall that for finite dimensional spaces $V \cong V^{**}$). We have a canonical isomorphism by choosing a basis $\{e_i\}_{i=1}^n$ for V, and bijecting between the element $\varphi \in |\Omega^n(V)|$ such that $\varphi(e_1, \dots, e_n) = 1$ with the Haar measure normalized such that it has the value 1 on the parallelogram spanned by the vectors $\{e_i\}_{i=1}^n$. This is independent of choice of basis since given a different basis both elements would be multiplied by the same factor of $|\det(M)|$, where M is the change of basis matrix with respect to these two bases. $\square$

Exercise 5.3.7. *Show the following:*

(1) $|L \otimes M| = |L| \otimes |M|$ *for two one dimensional vector spaces* L *and* M*.*
(2) $|\Omega^{\mathrm{top}}(V)| \simeq \mathrm{Dens}(V)$*.*
(3) If $W \subseteq V$ *then* $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V/W) \cong \mathrm{Haar}(V)$*.*
(4) If $W \subseteq V$ *then* $\Omega^{\mathrm{top}}(V) \cong \Omega^{\mathrm{top}}(W) \otimes \Omega^{\mathrm{top}}(V/W)$*.*
(5) If $F = \mathbb{R}$*, then* $\mathrm{Ori}(V) \cong \mathrm{Ori}(W) \otimes \mathrm{Ori}(V/W)$*.*

5.4. **Generalized functions supported on a subspace.** Let $W \subseteq V$ be linear spaces. We showed that over a non-archimedean local field F we have that $\mathrm{Dist}_W(V) = \mathrm{Dist}(W)$, and for $F = \mathbb{R}$ we described $\mathrm{Dist}_W(V)$ for the case where $V = \mathbb{R}^n$ and $W = \mathbb{R}^k$. The goal now is to describe distributions on V supported on W for any real linear spaces $W \subseteq V$. Recall we have defined a (non-exhausting) filtration $V_m(W)$ on $C_c^\infty(V)$ by

$$V_m(W) = \{f \in C_c^\infty(V) | \forall i \in \mathbb{N}_0^{n-k} \text{ where } |i| \leq m \text{ it holds that } \frac{\partial^i f}{(\partial x)^i}|_W = 0\}.$$

where $\dim(V) = n$ and $\dim(W) = k$. We then defined $F_m(W) \subseteq \mathrm{Dist}_W(V)$ by

$$F_m(W) = (C_c^\infty(V)/V_m(W))^* := \{\xi \in \mathrm{Dist}(V) | \langle \xi, f \rangle = 0 \text{ for any } f \in F_W^m(V)\}.$$

Note that we have that $F_m(W) = V_m(W)^\perp$ where $V_2^\perp := (V_1/V_2)^*$ for two vector spaces $V_2 \subseteq V_1$. We want to describe $F_m(W)/F_{m-1}(W)$ in canonical terms, that is such that the isomorphism will respect diffeomorphisms of V which preserve W.

Theorem 5.4.1. *We have an isomorphism of vector spaces which commutes with diffeomorphisms of* V *which preserve* W*:*

$$F_m(W)/F_{m-1}(W) \cong_{\mathrm{can}} C_c^\infty(W, \mathrm{Sym}^m(W^\perp))^* \simeq \mathrm{Dist}(W) \otimes \mathrm{Sym}^m(V/W).$$

The proof of the theorem is based on the following lemma:

Lemma 5.4.2. $F_m(W)/F_{m-1}(W) \cong (V_{m-1}(W)/V_m(W))^*$.

Proof. For any $\phi \in F_m(W)$, the restriction $\phi_{|V_{m-1}(W)}$ vanishes on $V_m(W)$, and we send it to the induced functional on $V_{m-1}(W)/V_m(W)$ which we denote by $\widetilde{\phi}$. This is an injective morphism, since if $\widetilde{\phi} = 0$ then $\phi|_{V_{m-1}(W)} = 0$ so $\phi \in F_{m-1}(V)_W$. Surjectivity follows from the Hahn-Banach theorem in the following way: any $\varphi \in (V_{m-1}(W)/V_m(W))^*$ can be extended to $\widetilde{\varphi} \in (C_c^\infty(V)/V_m(W))^* = F_m(W)$. Therefore $[\widetilde{\varphi}] + F_{m-1}(W) \mapsto \varphi$. $\square$

Hence, in order to prove the theorem it will be sufficient to prove that

$$V_{m-1}(W)/V_m(W) \cong C_c^\infty(W, \mathrm{Sym}^i(W^\perp)).$$

For this we do the natural thing- we attach to $f \in V_{m-1}(W)/V_m(W)$ its i-th derivatives. Explicitly, we define:

$$\Phi(f)(w)(v_1, \ldots, v_i) = \partial_{v_1} \ldots \partial_{v_i} f(w).$$

It is well defined as f vanishes identically on W, so this form vanishes on all the tangential derivatives. It is injective since if $\Phi(f) = 0$ then f vanishes with all of its derivatives up to degree i, so it is in $V_m(W)$.

Exercise 5.4.3.

(1) *Finish the proof of the lemma - show that Φ is onto, hence an isomorphism.*
(2) *Show that the isomorphism $F_m(W)/F_{m-1}(W) \cong_{\mathrm{can}} C_c^\infty(W, \mathrm{Sym}^m(W^\perp))^*$ is invariant with respect to diffeomorphism of (V, W).*
(3) *Find $\xi \in \mathrm{Dist}(V \backslash W)$ such that there is **no** $\eta \in \mathrm{Dist}(V)$ such that $\eta|_{V \backslash W} = \xi$. That is, show that the natural map $\mathrm{Dist}(V) \to \mathrm{Dist}(V \backslash W)$ is not onto.*

To get a similar result for generalized functions, we twist by the one dimensional space of Haar measures:

$$F_m(W)/F_{m-1}(W) \cong C_c^\infty(W, \mathrm{Sym}^m(W^\perp))^* = C^{-\infty}(W, \mathrm{Sym}^m(W^\perp) \otimes \mathrm{Haar}(W)).$$

Take $G_m(W) = F_m(W) \otimes \mathrm{Haar}(W)^* \subseteq C^{-\infty}(V)$. We get by the compatibility of tensor product and quotient the following:

$$\begin{aligned} G_m(W)/G_{m-1}(W) &\cong C^{-\infty}(W, \mathrm{Sym}^m(W^\perp) \otimes \mathrm{Haar}(W)) \otimes \mathrm{Haar}(V)^* \\ &\cong C^{-\infty}(W, \mathrm{Sym}^m(W^\perp) \otimes \mathrm{Haar}(W) \otimes \mathrm{Haar}(V)^*). \end{aligned}$$

The next exercise shows that $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V)^*$ can be presented in a simpler manner:

Exercise 5.4.4. *Let $W \subseteq V$.*

(1) *Show that* $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V/W) \cong_{\mathrm{can}} \mathrm{Haar}(V)$*.*
(2) *Show that* $\mathrm{Haar}(V^*) \cong_{\mathrm{can}} \mathrm{Haar}(V)^*$*.*
(3) *Conclude that* $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V)^* \cong_{\mathrm{can}} \mathrm{Haar}(W^\perp)$*.*

We arrive at the following corollary, yielding the desired description for generalized functions.

Corollary 5.4.5. *By the above argument it follows that:*

$$G_m(W)/G_{m-1}(W) \cong C^{-\infty}(W, \mathrm{Sym}^m(W^\perp) \otimes \mathrm{Haar}(W^\perp)).$$

6. Manifolds

After understanding generalized functions on vector spaces, we move to understand generalized functions on spaces which locally look like vector spaces. For this we define the notion of a manifold.

Definition 6.0.1. *Let X be a topological space.*

1. *A cover $\{U_i\}_{i\in I}$ is called* locally finite*, if for any $x \in X$ there is a neighborhood V such that $V \cap U_i \neq \varnothing$ only for finitely many $i \in I$.*
2. *X is called* paracompact*, if any open cover has a locally finite refinement.*
3. *X is called a* topological manifold *if X is locally homeomorphic to $\mathbb{R}^n$ and is Hausdorff and paracompact.*

Exercise 6.0.2.

1. *Find a space X which is locally homeomorphic to $\mathbb{R}^n$ at every point and is paracompact but is not Hausdorff.*
2. *Find a space which is Hausdorff, locally isomorphic to $\mathbb{R}^n$ but is not paracompact.*

We now give a definition of a smooth manifold which is different than the usual definition in differential topology and uses sheaves of functions.

Definition 6.0.3. *A* sheaf of (K-valued) functions *$\mathcal{F}$ on a topological space X is an assignment $U \mapsto \mathcal{F}(U) \subseteq \{f : U \to K |\, f$ is continuous$\}$ for every open $U \subset X$ such that:*

1. *$\mathcal{F}(U)$ is an algebra with unity.*
2. *If $f \in \mathcal{F}(U)$ and $V \subset U$ then the restricted function $f|_V$ belongs to $\mathcal{F}(V)$.*
3. *(Gluing) For every open cover $U = \bigcup\limits_{i\in I} U_i$, and every collection of functions $\{f_i \in \mathcal{F}(U_i)\}_{i\in I}$ such that $f_i|_{U_i\cap U_j} \equiv f_j|_{U_i\cap U_j}$ for any $i, j \in I$, there exists $f \in \mathcal{F}(U)$ s.t. $f|_{U_i} \equiv f_i$ for any $i \in I$.*

Note that a sheaf of functions is a sheaf. The identity axiom is automatic.

Example 6.0.4. *Continuous or smooth functions on a space X form a sheaf of functions. So do locally constant functions.*

Definition 6.0.5. *A space with functions is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ is a sheaf of functions on X. A morphism of spaces with functions $\varphi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a continuous map $\varphi : X \to Y$ such that for any open $U \subset Y$ and any function $f \in \mathcal{G}(U)$, the composition $f \circ (\varphi|_U)$ lies in $\mathcal{F}(\varphi^{-1}(U)$.*

In the language of sheaves, the composition with φ defines $\varphi^\# : \mathcal{G} \to \varphi_*\mathcal{F}$

Definition 6.0.6. *A* smooth manifold *is a space with a sheaf of functions $(X, C^\infty(X))$ such that X is a topological manifold and for every point $x \in X$ there is an open neighborhood U such that $(U, C^\infty(X)|_U) \simeq (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ as sheaves of functions.*

Remark 6.0.7. *The usual definition of manifolds adds an atlas to the structure of X, that is an open cover $X = \bigcup\limits_{i\in I} U_i$ with diffeomorphisms $\phi_i : U_i \to \mathbb{R}^n$. We also demand that $\phi_i \circ \phi_j^{-1}$ is differentiable, so it seems like an additional demand* *with respect to the definition above. Alas, further rumination shows that a pair of isomorphisms* $\varphi_i : (U_i, C^\infty(U_i)) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ *and* $\varphi_j : (U_j, C^\infty(U_j)) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ *implies that the following composition is an isomorphism:*

$$\left(\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j}\right)^{\#} : (\mathbb{R}^n, C^\infty(\mathbb{R}^n))|_{\varphi_j(U_i\cap U_j)} \to \left(\varphi_i \circ \varphi_j^{-1}\right)_* (\mathbb{R}^n, C^\infty(\mathbb{R}^n))|_{\varphi_i(U_i\cap U_j)}.$$

In particular, by the next exercise we can deduce that $\varphi_i \circ \varphi_j^{-1}|_{U_i\cap U_j}$ *is smooth and is a diffeomorphism. Therefore Definition 6.0.6(2) is equivalent to the usual definition of a smooth manifold.*

Exercise 6.0.8.

(1) *Show that* $C^\infty(\mathbb{R}^n, \mathbb{R}^k) = \{f : \mathbb{R}^n \to \mathbb{R}^k : \, f^*(\mu) \in C^\infty(\mathbb{R}^n)\, \forall \mu \in C^\infty(\mathbb{R}^k)\}$.
(2) *Let* M *and* N *be smooth manifolds. Show that 6.0.6(1) is equivalent to the usual definition of a morphism of smooth manifolds. That is, that a map* $f : M \to N$ *is a smooth map of manifolds* $\iff$ *it is a morphism of ringed spaces (where the sheaf is a sheaf of smooth functions).*

Remark 6.0.9. *Note that by a theorem of Whitney every* n*-dimensional manifold can be embedded in* $\mathbb{R}^{2n+1}$. *Thus we can always think about smooth manifolds sitting in* $\mathbb{R}^N$ *for* N *large enough.*

6.1. **Tangent space of a manifold.** There are several equivalent ways to define the tangent space to a smooth manifold M at a point $x \in M$. We first give a categorical definition and then construct several objects which satisfy this definition.

Definition 6.1.1. *We denote by* ptMan *the category of smooth pointed manifolds, that is the objects are pairs consisting of a smooth manifold* M *and a point* $x \in M$ *and the morphisms are smooth maps of manifolds which preserve the distinguished points.*

Definition 6.1.2. *A tangent space is a functor* $\mathrm{Tan} : \mathrm{ptMan} \to \mathrm{Vect}$ *from pointed smooth manifolds to vector spaces which satisfies the following conditions:*

(1) *The restriction of* Tan *to the subcategory* $Vect \subset ptMan$ *is the identity functor.*
(2) *If* $f, g : (\mathbb{R}, 0) \to (\mathbb{C}, 0)$ *satisfy* $f'(0) = g'(0)$ *then* $\mathrm{Tan}(f) = \mathrm{Tan}(g)$.
(3) *If* $\varphi : U \hookrightarrow M$ *is an open embedding, then* $\mathrm{Tan}(\varphi)$ *is an isomorphism.*

There are several structures that satisfy the above conditions:

(1) The space of all smooth paths $T_x(M) := \{\gamma : ((-1,1), 0) \to (M, x)\}$ modulo the relation $\gamma_1 \sim \gamma_2 \iff$ there exists a neighborhood U of x and an isomorphism $\phi : U \to \mathbb{R}^n$ s.t. $(\phi \circ \gamma_1)'(x) = (\phi \circ \gamma_2)'(x)$. It is easy to check that this definition does not depend on the choice of (ϕ, U).
(2) The space of derivations

$$T_x(M) = \{\partial : C^\infty(M) \to \mathbb{R} : \partial \text{ is linear}, \partial(f \cdot g) = \partial f \cdot g(x) + f(x) \cdot \partial g\}.$$

(3) Define $m_x := \{f \in C^\infty(M) : f(x) = 0\}$, and take $T_x(M) := (m_x/m_x^2)^*$.

Exercise 6.1.3. *Show the constructions of the tangent space given above are equivalent.*

Definition 6.1.4. *Let $\phi : M \to N$ be a smooth map. The* differential *of ϕ at $x \in M$ is the map $d_x\phi : T_x(M) \to T_{\phi(x)}(N)$ defined by $d_x(\phi)(\gamma) := \phi \circ \gamma$ for an equivalence class of paths $[\gamma] \in T_x(M)$.*

Exercise 6.1.5.

(1) Show the differential is well defined, i.e. it does not depend on the representative $\gamma \in [\gamma]$.
(2) Show that given manifolds M, N, and K and maps $\phi : M \to N$ and $\psi : N \to K$, the differentials satisfy $d_x(\psi \circ \phi) = d_{\phi(x)}(\psi) \circ d_x(\phi)$.

6.2. **Types of maps between smooth manifolds.**

Definition 6.2.1. *Let $\phi : M \to N$ be a smooth map between smooth manifolds.*

(1) ϕ is an immersion *if $d_x\phi$ is injective.*
(2) ϕ is a submersion *if $d_x\phi$ is surjective.*
(3) ϕ is a local isomorphism *or* étale *if $d_x\phi$ is an isomorphism.*
(4) ϕ is an embedding *if it is an immersion and defines a homeomorphism $M \cong \phi(M)$.*
(5) ϕ is a proper map *if for every compact $K \subset N$, the preimage $\phi^{-1}(K)$ is compact. In particular, in that case all the fibers of ϕ are compact in M.*
(6) ϕ is a covering map *if for every $x \in N$ there exists a neighborhood $U \subseteq N$, such that $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is locally a diffeomorphism, and $\phi^{-1}(U) \cong U \times D$ for some discrete set D.*

Example 6.2.2.

(1) Let $\phi : [-1, 1] \to \mathbb{R}^2$ be a smooth path that slows to a stop at $\phi(0) = (0, 0)$, but spends no time at $(0, 0)$. That is, $\phi'(0) = (0, 0)$, but $\phi(x) \neq 0$ for all $x \neq 0$ in some neighborhood $[-\varepsilon, \varepsilon]$ of 0. Such a ϕ is locally injective at 0, but since $d_0\phi = 0$ it is not an immersion at 0.
(2) An immersion is not necessarily one-to-one. As an example, consider a self-intersecting path $\phi : \mathbb{R} \to \mathbb{R}^2$ with constant speed.
(3) Let L and D be finite dimensional linear spaces. The differential of a linear map $\phi : L \to V$ is ϕ itself. Thus, a one-to-one ϕ will be an immersion, an onto ϕ will be a submersion, and an isomorphism of linear space will be an étale map.

Exercise 6.2.3. *Let M and N be smooth manifolds.*

(1) Find a map $\phi : M \to N$ which is an injective immersion, but is not an embedding.

(2) *Show that every proper map which is an injective immersion is a closed embedding.*
(3) *Show that a proper map which is étale is a covering map, and that a covering map with finite fibers is proper and étale.*

Definition 6.2.4. *Let X and Y be topological spaces. A* fiber bundle *is a map $p : X \to Y$, such that for every $y \in Y$ there exists a neighborhood $U \subseteq Y$ such that $p^{-1}(U) \simeq U \times Z$ for $U \subseteq Y$ for some topological space Z.*

Exercise 6.2.5. *Show that a proper submersion is a fiber bundle.*

6.3. **Analytic manifolds and vector bundles.** We would like to be able to talk about manifolds for a general local field. In order to do so, for a non-archimedean local field F we introduce the notion of an analytic F-manifold.

Definition 6.3.1. *Let F be a non-archimedean local field. An analytic F-manifold is a topological space M which is locally isomorphic to $\mathcal{O}_F^n$ together with the sheaf of functions*

$$\mathrm{An}(U) = \{f : U \to F : \forall x \in U, \exists r > 0 \text{ s.t. } f_{|B_r(x)}(y) = \sum_{\vec{k} \in \mathbb{N}^n} a_{\vec{k}}(x-y)^{\vec{k}}\},$$

where $B_r(x)$ is the ball of radius r around x, $\vec{k}$ is a multi-index, and $(x-y)^{\vec{k}} = \prod_{i=0}^{n}(x_i - y_i)^{k_i}$.

Remark 6.3.2. *By Exercise 4.6.8 there is no need to use partition of unity for F-analytic manifolds.*

Example 6.3.3. *There exist singular analytic manifolds, and any singular affine algebraic variety is an example for such a manifold.*

Definition 6.3.4. *Let M be a smooth manifold or a p-adic analytic manifold. A real vector bundle over M is a tuple (E, p) where E is a topological space and $p : E \to M$ is a continuous surjection such that:*

(1) *For every $x \in M$ we have a structure of a finite dimensional real vector space on $p^{-1}(x) = V_x$.*
(2) *For every $x \in M$ there exists an open $x \in U$ and a local trivialization $\varphi_U : V_x \times U \to p^{-1}(U)$ where φ_U is a homeomorphism (or diffeomorphism if M is a real smooth manifold) and $p \circ \varphi_U(v, x) = x$ for all $v \in V_x$.*
(3) *The maps $v \mapsto \varphi_U(v, x)$ are linear isomorphisms.*

If $E \simeq V \times M$ we say (E, p) is a trivial bundle over M.
If $\dim V_x = 1$ for all $x \in M$ we say (E, p) is a line bundle over M.

Exercise 6.3.5. *It is known that the Mobius strip M is not homeomorphic to the (finite) cylinder $I \times S^1$. By extending each segment I of M to $\mathbb{R}$, we can define a* *vector bundle E over the manifold S^1. This way, points of E are such that the fiber over $\theta \in S^1$ is a line in $\mathbb{R}^3$ which intersects the z-axis with angle $0.5 \cdot \theta$. Define the vector bundle above rigorously and show it is not diffeomorphic to the trivial bundle $S^1 \times \mathbb{R}$.*

Example 6.3.6. *The tangent bundle of $M = S^1$ is $TS^1 \simeq S^1 \times \mathbb{R}$. The tangent space at any point is one-dimensional, and changes smoothly as we move along the circle. However, on $M = S^2$ the tangent bundle is not isomorphic to $S^2 \times \mathbb{R}^2$. This happens since every vector field on S^2 vanishes at some point. (Hairy ball theorem).*

Definition 6.3.7. *Let (M, E) be a k-dimensional real vector bundle and π be its projection. Given trivializing neighborhoods U and V, and trivializations $\varphi_U : U \times \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(U)$ and $\varphi_V : V \times \mathbb{R}^k \xrightarrow{\sim} \pi^{-1}(V)$, one can consider $\varphi_V^{-1} \circ \varphi_U : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$. We can then write $\varphi_V^{-1} \circ \varphi_U(x, v) = (x, g_{U,V}(v))$ where $g_{U,V} \in \mathrm{GL}(\mathbb{R}^k)$. The maps $g_{U,V}$ are called* transition functions.

Notice that the set of transition functions $g_{U,V}$, satisfy the cocycle conditions

$$g_{U,U}(x) = Id \text{ and } g_{U,V}(x) g_{V,W}(x) = g_{U,W}(x).$$

Conversely, given a fiber bundle (E, X, π) of degree k with a transition map in $\mathrm{GL}(\mathbb{R}^k)$ abiding the cocycle condition which acts in the standard way on the fiber $\mathbb{R}^k$, there is an associated a vector bundle. This is sometimes taken as the definition of a vector bundle.

Proposition 6.3.8. *Given a manifold M, vector bundles $\{(E_i, p_i)\}_{i=1}^n$ each of which with fiber of constant dimension m_i over it, and a functor $F : \mathrm{Vect}^n \to \mathrm{Vect}$, we can construct a vector bundle $(F(E_1, \ldots, E_n), q)$ over M.*

Proof. First, take a cover $\{U_\alpha\}_{\alpha \in I}$ of M which is a local trivialization of E (that is, $p^{-1}(U_\alpha) \simeq V \times U_\alpha$). Define the total space $F(E)$ over each U_α by $F(V) \times U_\alpha$, where the surjection q will be projecting to M, and glue every two pieces $q^{-1}(U_\alpha)$ and $q^{-1}(U_\beta)$ by setting $(v, x) \sim (g_{\alpha,\beta}(v), x)$ for every $x \in U_\alpha \cap U_\beta$ and $v \in V$, where $g_{\alpha,\beta} = F(\varphi_{U_\beta}^{-1} \varphi_{U_\alpha})$. Finally, note that for any two elements of the cover $g_{\alpha,\beta}^{-1} = g_{\beta,\alpha}$, and that in order for our construction to be well defined we need to show the cocycle condition, namely that $g_{\beta,\gamma} g_{\alpha,\beta} = g_{\alpha,\gamma}$ when restricted to triple intersections. This holds since

$$g_{\beta,\gamma} g_{\alpha,\beta} = F(\varphi_{U_\gamma}^{-1} \varphi_{U_\beta}) F(\varphi_{U_\beta}^{-1} \varphi_{U_\alpha}) = F(\varphi_{U_\gamma}^{-1} \varphi_{U_\alpha}) = g_{\alpha,\gamma}.$$

Note that if we want $F(E)$ to have a smooth structure we need to demand that F preserves smooth maps. □

Example 6.3.9. *Let E_1 and E_2 be two vector bundles over M. The direct sum $E_1 \oplus E_2$ is defined by applying our construction above to the direct sum functor $\bigoplus : \mathrm{Vect}^2 \to \mathrm{Vect}$.*

Exercise 6.3.10. *Find two non-isomorphic bundles E and E', such that $E \oplus F \cong E' \oplus F$ for a bundle F (**Hint**: use vector bundles over S^2).*

Definition 6.3.11. *Let M be a manifold.*

(1) *The tangent bundle of M is the disjoint union of its tangent spaces $TM = \bigcup_{x\in M} \{x\} \times T_x M$.*
(2) *Given a submanifold $N \subseteq M$, and an embedding $i : N \to M$, we define the* normal bundle *to N in M to be $\mathrm{N}_N^M := i^*(TM)/TN$, where i^* is the pullback of the bundle TM to N. Similarly, the* conormal bundle *to N in M is $\mathrm{CN}_N^M := (\mathrm{N}_N^M)^*$.*

Example 6.3.12. *For the sphere $N = S^2 \subset \mathbb{R}^3 = M$, the normal bundle at a point is the normal line to it (i.e. the line passing at the point and at zero). It is diffeomorphic to the trivial bundle on N.*

Definition 6.3.13. *For vector bundles E_1, E_2 over a manifold (smooth or F-analytic), we define the following:*

(1) E_1^*.
(2) $E_1 \oplus E_2$.
(3) $E_1 \otimes E_2$.
(4) *For an embedding $\varphi : E_1 \hookrightarrow E_2$, we define E_2/E_1.*
(5) $\bigwedge^k(E_1)$, $\mathrm{Sym}^k(E_1)$.
(6) *We define the density bundle of E_1 by $\mathrm{Dens}(E_1)$.*

Definition 6.3.14. *Let M be either a smooth manifold or an F-analytic manifold, we define its* density bundle *by $\mathrm{Dens}(M) = |\Omega^{\mathrm{top}}(TM)|$, that is the density bundle of its tangent bundle.*

6.4. **Sections of a bundle.** A set theoretic *section* of a function $f : X \to Y$ is a function $g : Y \to X$ s.t. $g \circ f = \mathrm{id_X}$.

Example 6.4.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection $f(x, y) = x$. One example of a section is $g(x) := (x, sinx)$.*

In many case sections of bundles give rise to important concepts:

- A section of the tangent bundle of a manifold is a *vector field*.
- A section of the k-th exterior power of the cotangent bundle of a manifold is a *differential form* of degree k.
- A section of the density bundle is a *density* on the manifold.
- A section of the orientation bundle is a choice of an *orientation* on the manifold.

Exercise 6.4.2.

(1) *Show the every manifold has a Riemannian metric, i.e , an inner product on tangent spaces*

$$\langle\cdot,\cdot\rangle_p : T_pM \times T_pM \to \mathbb{R}$$

which varies smoothly.

(2) *Let M be a smooth n-dimensional Riemannian manifold, that is a smooth real manifold with a Riemannian metric. Construct explicitly a density over M, that is a smooth section of the density bundle over M. The density should respect coordinate changes, and be the standard density when M is a linear space with the standard inner product.*

Remark 6.4.3. *We do not always have non-zero top differential forms on a manifold M, and the Mobius strip is an example of a manifold with no non-zero top differential form. However, we can always find a non-zero density on M. Since with a density we can define a measure on the manifold, we can define integration over manifolds.*

6.5. **Another description of vector bundles.**

Definition 6.5.1. *Let V be a finite dimensional vector space and X a topological space.*

(1) *We define the* constant sheaf $\underline{V}_X$ *to be the sheafification of the constant presheaf, which assigns to every open set in X the vector space V.*

(2) *We say that a sheaf $\mathcal{F}$ over X is* locally constant *if for every $x \in X$ there exists an open $x \in U_x$ and a finite dimensional vector space V_x such that $\mathcal{F}_{|U_x} \simeq \underline{V_x}_{U_x}$.*

Exercise 6.5.2. *Let V be a finite dimensional vector space and X a topological space.*

(1) *Show that $\underline{V}_X(U)$ consists of the locally constant functions from U to V.*

(2) *Show that if X is a σ-compact ℓ-space then every locally constant sheaf $\mathcal{F}$ such that $\mathcal{F}_x \simeq \mathcal{F}_y$ for all $x, y \in X$ is isomorphic to the constant sheaf.*

Up to now we have used the Grothendieck definition of a sheaf. In some situations the following definition is more useful.

Definition 6.5.3. *A* Leray *sheaf on X is a pair (E, p) such that E is a topological space and $p : E \to X$ is a homeomorphism locally in E, i.e. every point in E has an open neighborhood U such that $p(U)$ is open and $p|_U$ defines a homeomorphism $U \simeq p(U)$.*

Theorem 6.5.4. *The category of Leray sheaves is equivalent to the category of Grothendieck sheaves.*

Proof. Given a Leray sheaf (E, p) we define a Grothendieck sheaf by

$$\mathcal{F}(U) := \{s : U \to p^{-1}(U) : f \text{ is continuous and } p \circ s = \mathrm{Id}_U\}$$

with the obvious restriction maps.
For the other direction, given a Grothendieck sheaf $\mathcal{F}$, we define $E = \bigsqcup_{x \in X} \mathcal{F}_x$ with the natural projection map $p : E \to X$. The basis of the topology of E is given by $U_{s,V} = \{(x, (s)_x) : x \in V\}$ where $V \subseteq X$ is open and $s \in \mathcal{F}(V)$. □

Exercise 6.5.5. *Complete the proof by showing that this is indeed an equivalence of categories.*

Exercise 6.5.6.

(1) Show that covering spaces correspond to locally constant sheaves, and that a covering space is trivial when it corresponds to a constant sheaf.
(2) Give an example of a locally constant sheaf arising from a covering space which is not constant.

7. Distributions on analytic manifolds and on smooth manifolds

Definition 7.0.1. *Let E be an F-analytic line bundle over an F-analytic manifold X. Define a real vector bundle $|E|$ as follows. As a set define $|E| := \{(x, v) : x \in X, v \in |E_x|\}$ and define a topology by giving $\mathbb{C}$ the discrete topology, so locally $E|_U \simeq U \times F$ and $|E|\,|_U \simeq U \times |F| \simeq U \times \mathbb{C}$. Hence, a base for the topology is $V_{i,U,\alpha} = \varphi_i(U \times \{\alpha\})$ where $\varphi_i : U \times \mathbb{C} \to |E|\,|_U$ and $\alpha \in \mathbb{C}$.*

Remark 7.0.2. *Note that $\widetilde{p} : |E| \to X$ is a local homeomorphism as $V_{i,U,\alpha} \simeq U$ as a topological space. Hence $\widetilde{p}$ is a Leray sheaf. Its corresponding Grothendieck sheaf is $\mathcal{F}(U) := \{$continuous sections $U \to \widetilde{p}^{-1}(U)\}$. This is a locally constant sheaf $\underline{\mathbb{C}}_X$ over X.*

Definition 7.0.3. *We can now define the density bundle over an F-analytic manifold X in two ways:*
Def 1 (Leray): $\mathrm{Dens}(X) := |\Omega^{\mathrm{top}}(X)|$.
Def 2 (Grothendieck):

$$\mathrm{Dens}(X)(U) := \{\mu \in \mathrm{Measures}(U) | \forall \varphi \in \mathcal{O}_F^n \to U, \textit{there exists } f \in C^\infty(\mathcal{O}_F^n) \textit{ such that } \mu = \varphi_*(f \cdot \mathrm{Haar})\}.$$

Lemma 7.0.4. *Let $\varphi : F^n \to F^n$ be an analytic diffeomorphism, let $f \in C_c(F^n)$ and let μ be a choice of a Haar measure on F^n. Then*

$$\langle \mu, f \rangle =: \int f dx = \int (f \circ \varphi)\, |det(D_x\varphi)|\, dx.$$

Exercise 7.0.5. *Show that the above definitions are equivalent.*

7.1. **Smooth sections of a vector bundle.** In this subsection we assume that $F = \mathbb{R}$ and we are dealing with smooth manifolds.

Definition 7.1.1. *We define smooth functions on a manifold M with compact support and values in a vector bundle (E, π) by:*

$$C_c^\infty(M, E) := \{f : M \to E : \pi \circ f = Id_M \text{ and } \exists K \text{ compact such that } f|_{K^C}(m) = (m, 0)\}.$$

Recall that $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) = \underrightarrow{lim} C_{K_m}^\infty(\mathbb{R}^n, \mathbb{R}^k)$ where K_m is an increasing sequence of compact sets such that $\bigcup_{n=1}^{\infty} K_m = \mathbb{R}^n$. We define a topology on $C_c^\infty(M, E)$ using the topology on $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$:

Case 1- The trivial case: $M \simeq \mathbb{R}^n$ and $E \simeq \mathbb{R}^n \times \mathbb{R}^k$ with the projection to the first component. Note that continuous sections from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^k$ are just functions in $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$. Hence we give $C_c^\infty(M, E)$ the topology of $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$.

Exercise 7.1.2. *Show that the above definition is well defined, i.e. it does not depend on the isomorphism $M \simeq \mathbb{R}^n$ and $E \simeq \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$. In other words, show that:*

1. *Given a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ it induces a homeomorphism $\varphi^* : C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k)$ via precomposition.*
2. *Given a smooth map $\psi \in C^\infty(\mathbb{R}^n, \mathrm{GL}_k(\mathbb{R}))$ we have that $\psi_* : C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^k)$ by $\psi_*(f) = \psi \circ f$ is a homeomorphism.*

Case 2- General case: We can choose trivializing $\{U_i\}_{i \in I}$ such that $M = \bigcup_{i \in I} U_i$ where $\varphi_i : U_i \xrightarrow{\simeq} \mathbb{R}^n$ and $\psi_i : E|_{U_i} \xrightarrow{\simeq} \mathbb{R}^n \times \mathbb{R}^k$. We have a surjective map

$$\varphi : \bigoplus_{i \in I} C_c^\infty(U_i, E|_{U_i}) \twoheadrightarrow C_c^\infty(M, E)$$

by summation where surjectivity follows from partition of unity. We define the quotient topology on $C_c^\infty(M, E)$ according to the map φ, that is, a set $U \subseteq C_c^\infty(M, E)$ is open if $\varphi^{-1}(U)$ is open in $\bigoplus_{i \in I} C_c^\infty(U_i, E|_{U_i})$, where the latter is endowed with the direct sum topology.

Proposition 7.1.3. *The topology on $C_c^\infty(M, E)$ is well defined. That is, the definition does not depend on the choice of the cover $\{U_i\}_{i \in I}$ of M.*

Proof. We need to show that given a different cover $\{V_\beta\}_{\beta \in J}$ of M which locally trivializes M and E, we get the same topology.

Consider the cover $\{W_{\alpha,\beta}\}$ for $W_{\alpha,\beta} = U_\alpha \cap V_\beta$ which refines both covers. We need to show that for the addition map,

$$\bigoplus_{\alpha \in I} \bigoplus_{\beta \in J} C_c^\infty(W_{\alpha,\beta}, E_{|_{W_{\alpha,\beta}}}) \xrightarrow{+} \bigoplus_{\alpha \in I} C_c^\infty(U_\alpha, E_{|_{U_\alpha}})$$

a set in the range is open if and only if its preimage is open, where $W_{\alpha,\beta} \subseteq U_\alpha \simeq \mathbb{R}^n$ and $E_{|_{W_{\alpha,\beta}}} \simeq E_{|_{U_\alpha}} \simeq \mathbb{R}^k$. In order to show the above, it is enough to handle each case $\bigoplus_{\beta\in J} C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k) \xrightarrow{+} C_c^\infty(U_\alpha,\mathbb{R}^k) \simeq C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$ separately, since in the direct sum topology a set is open if all the injections $D_i \hookrightarrow \bigoplus D_i$ are continuous (we furthermore assume our open sets are convex).

Given a basic open set $U_{(L_m,\epsilon_m,B_m)} \subseteq C_c^\infty(U_\alpha,\mathbb{R}^k)$ where L_m are mixed differentiations, $\epsilon_m \in \mathbb{R}_{>0}$ and B_m are compact sets such that $\bigcup_{m=1}^{\infty} B_m = \mathbb{R}^n$, it is of the form $U_{(L_m,\epsilon_m,B_m)} = \sum_{m\in\mathbb{N}} V_{L_m,\epsilon_m,B_m}$, where

$$V_{L_m,\epsilon_m,B_m} = \left\{ f \in C^\infty(\mathbb{R}^n,\mathbb{R}^k) : \mathrm{supp}(f) \subseteq B_m, \sup_{x\in\mathbb{R}^n} ||L_m(f)|| < \epsilon_m \right\}.$$

Now, take a finite sum $\sum f_\beta \in +^{-1}(U_{\{L_m,\epsilon_m,B_m\}})$ for $\sum f_\beta = f = \sum_{i=0}^{l} f_{m_i}$ and $f_{m_i} \in V_{L_{m_i},\epsilon_{m_i},B_{m_i}}$. Let $f_\beta = pr_\beta(f)$ be the projection of f into $C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)$, and define $N = \#\{\beta : pr_\beta(f) \neq 0\}$ and $\epsilon'_{m_i} = \frac{\epsilon_{m_i} - \sup ||L_{m_i}(f_{m_i})||}{N}$ and set $\epsilon'_m = \frac{\epsilon_m}{N}$ if $m \neq m_i$ for all $0 \le i \le l$. For $B'_{m,\beta} \subseteq W_{\alpha,\beta}$, compact sets which exhaust $W_{\alpha,\beta}$ and such that $B'_{m,\beta} \subseteq B_m$, the sets $U_{(L_m,\epsilon'_m,B'_{m,\beta})}$ are basic open sets in each $C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)$, and their direct sum is open in the direct sum. Now, we claim that,

$$f \in \bigoplus_{\beta: f_\beta \neq 0} f_\beta + U_{(L_m,\epsilon'_m,B'_{m,\beta})} \subseteq +^{-1}(U_{(L_m,\epsilon_m,B_m)}).$$

Given $g = \sum_{\beta: f_\beta\neq 0} g_\beta$ where $g_\beta \in U_{(L_m,\epsilon'_m,B'_{m,\beta})}$, then $g_\beta = \sum_{i_\beta=1}^{l_\beta} g_{\beta,i_\beta}$ where $g_{\beta,i_\beta} \in V_{L_{n_{i_\beta}},\epsilon'_{n_{i_\beta}},B'_{n_{i_\beta},\beta}}$.

Thus if $n_{i_\beta} = m_i$ for some i, we have $\sup_{x\in B_{m_i}} ||L_{m_i}(g_{\beta,m_i})|| < \epsilon'_{m_i} = \frac{\epsilon_{m_i} - \sup ||L_{m_i}(f)||}{N}$ implying that,

$$\begin{aligned}
\sup_{x\in B_{m_i}} \Big|\Big| \sum_{\beta: f_\beta\neq 0} L_{m_i}(f_{m_i,\beta} + g_{\beta,m_i}) \Big|\Big| &\le \sup_{x\in B_{m_i}} \Big|\Big| \sum_{\beta: f_\beta\neq 0} L_{m_i}(f_{m_i,\beta}) \Big|\Big| + \sum_{\beta: f_\beta\neq 0} \sup_{x\in B_{m_i}} ||L_{m_i}(g_{\beta,m_i})|| \\
&< \sup_{x\in B_{m_i}} \Big|\Big| L_{m_i}(f_{m_i}) \Big|\Big| + \sum_{\beta: f_\beta\neq 0} \Big(\frac{\epsilon_{m_i} - \sup ||L_{m_i}(f_{m_i})||}{N} \Big) \\
&= \epsilon_{m_i}.
\end{aligned}$$

Otherwise, if $n_{i_\beta} \neq m_i$ for all i, set $n' = n_{i_\beta}$, and using the requirement $\sup_{x\in B_{n'}} ||L_{n'}(g_{\beta,n'})|| < \frac{\epsilon_{n'}}{N}$ we note that:

$$\sup_{x\in B_{n'}} \Big|\Big| \sum_{\beta: f_\beta\neq 0} L_{n'}(g_{\beta,n'}) \Big|\Big| \le \sum_{\beta: f_\beta\neq 0} \sup_{x\in B_{n'}} ||L_{n'}(g_{\beta,n'})|| < N\frac{\epsilon_{n'}}{N} = \epsilon_{n'}.$$

This allows us to conclude that $f+g=\sum\limits_{\beta:f_\beta\neq 0}\sum\limits_{i=1}^{l} f_{m_i,\beta}+\sum\limits_{\beta:f_\beta\neq 0}\sum\limits_{i_\beta=1}^{l_\beta} g_{\beta,i_\beta}$ lie in $U_{(L_m,\epsilon_m,B_m)}=\sum\limits_{m\in\mathbb{N}} V_{L_m,\epsilon_m,B_m}$ for all such functions g, implying that the addition is continuous. For a less cumbersome approach, note that the embeddings $\bigoplus\limits_{\beta\in J} C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)\to C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$ are continuous (a cookie for the person who finds a quick proof for this), so it is enough to show that the addition map $\bigoplus\limits_{\beta\in J} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)\xrightarrow{+} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)\simeq\bigoplus\limits_{i=k}^{k} C_c^\infty(\mathbb{R}^n)$ is continuous. Since the domain has the direct sum topology, it is enough to check this for a finite direct sum, which follows by the continuity of addition in a topological vector space.
To show the map is open, it is enough to consider $\bigoplus\limits_{\beta\in J} C_{K,c}^\infty(W_{\alpha,\beta},\mathbb{R}^k)\xrightarrow{+} C_K^\infty(\mathbb{R}^n,\mathbb{R}^k)\simeq \bigoplus\limits_{j=1}^{k} C_K^\infty(\mathbb{R}^n)$, for every compact K, and since the domain has the direct sum topology and the basic open sets are finite sums of open sets in each coordinate, it is enough to show it for a finite direct sum $\bigoplus\limits_{i=1}^{m} C_{K,c}^\infty(W_i,\mathbb{R}^k)\xrightarrow{+}\bigoplus\limits_{j=1}^{k} C_K^\infty(\mathbb{R}^n)$ where $K\subset\bigcup\limits_{i=1}^{m} W_i$. Now, use partition of unity f_i, with $C_i=\mathrm{supp}(f_i)\subset W_i$ where $\sum\limits_{i=1}^{m} f_{i_{|K}}\equiv 1$ to get an onto map via the composition,

$$\bigoplus_{i=1}^{m} C_{K\cap C_i}^\infty(W_i,\mathbb{R}^k)\hookrightarrow\bigoplus_{i=1}^{m} C_{K,c}^\infty(W_i,\mathbb{R}^k)\xrightarrow{+} C_K^\infty(\mathbb{R}^n).$$

Since this is a continuous surjective map of Fréchet spaces, it must be open, implying that the addition is open since the embedding is continuous. □

We now give a different description of the topology of $C_c^\infty(\mathbb{R}^n)$. First observe that $f\in C(\mathbb{R}^n)$ is compactly supported if and only if fg is bounded for any $g\in C(\mathbb{R}^n)$. Now let $D\in\mathrm{Diff}(\mathbb{R}^n)$ be a differential operator on $C_c^\infty(\mathbb{R}^n)$. Define a seminorm $\|f\|_D$ by $\sup\limits_{x\in\mathbb{R}^n}|D(f)(x)|$.

Exercise 7.1.4. *The topology on $C_c^\infty(\mathbb{R}^n)$ can be defined by the seminorms $\|\ \|_D$.*

Definition 7.1.5. *Let M be a manifold and $D: C^\infty(M)\to C^\infty(M)$ be a map. We say that D is a differential operator on M if for any trivializing cover $\bigcup\limits_{i\in I} U_i=M$ and $\varphi_i: U_i\xrightarrow{\sim}\mathbb{R}^n$ we have $\varphi_i^{-1}\circ D\circ\varphi_i\in\mathrm{Diff}(\mathbb{R}^n)$. We denote the space of all differential operators on M by $\mathrm{Diff}(M)$.*

We would like to define differential operators from $C^\infty(M,E)$ to $C^\infty(M,E')$, which we denote by $\mathrm{Diff}(C^\infty(M,E),C^\infty(M,E'))$. As before we divide the definition into cases:

Case 1- the trivial case: Assume that $E \simeq M \times \mathbb{R}^k$ and $E' \simeq M \times \mathbb{R}^{k'}$. Then $\mathrm{Diff}(C^\infty(M,E), C^\infty(M,E')) \simeq \mathrm{Diff}(C^\infty(M)^k, C^\infty(M)^{k'})$ and the latter space is isomorphic as a vector space to the space of $k \times k'$ matrices with values in $\mathrm{Diff}(C^\infty(M))$.

Exercise 7.1.6. *Show that the definition of the space of differential operators* $\mathrm{Diff}(C^\infty(M,E), C^\infty(M,E'))$ *does not depend on the isomorphisms* $E \simeq M \times \mathbb{R}^k$ *and* $E' \simeq M \times \mathbb{R}^{k'}$.

Case 2- the general case: Let $A \in \mathrm{Hom}(C^\infty(M,E), C^\infty(M,E'))$. Then we say that $A \in \mathrm{Diff}(C^\infty(M,E), C^\infty(M,E'))$ if:

(1) For any $f_1, f_2 \in C^\infty(M,E)$ such that $f_1|_U = f_2|_U$, we have $Af_1|_U = Af_2|_U$.
(2) If $E'|_U$ is a trivialization then $A|_U \in \mathrm{Diff}(U, E|_U, E'|_U)$.

Definition 7.1.7 (Second definition to the topology on $C_c^\infty(M,E)$)**.** *For* $D \in \mathrm{Diff}(C^\infty(M,E), C^\infty(M,E))$ *define* $\|f\|_D = \sup_{x \in M} |D(f)(x)|$. *Define the topology on* $C_c^\infty(M,E)$ *via*

$$C_c^\infty(M,E) = \lim_{\leftarrow D} (C_c^\infty(M,E)_{\|\cdot\|_D}).$$

Exercise 7.1.8. *Given a manifold M and a vector bundle E over it show that the two definitions of the topology on $C_c^\infty(M,E)$ are equivalent (one defined via taking a cover of M and trivialization of E and the other through differential operators).*

7.2. **Distributions on manifolds.**

Definition 7.2.1. *Let M be a smooth or F-analytic manifold, and let E be a smooth vector bundle over it (in the case of an analytic manifold it has the discrete topology).*

(1) The space of distributional E-sections is defined to be $\mathrm{Dist}(M,E) := C_c^\infty(M,E)^*$.
(2) The space of generalized E-sections is defined to be $C^{-\infty}(M,E) = \mathrm{Dist}(M, E^* \otimes \mathrm{Dens}(M))$.

Although we do not have a natural injection from $C_c^\infty(M,E)$ to $C_c^\infty(M,E)^*$, we have a natural injection

$$i : C_c^\infty(M,E) \hookrightarrow C^{-\infty}(M,E)$$

as follows: let $\mu \in C_c^\infty(M, E^* \otimes \mathrm{Dens}(M))$ and $f \in C_c^\infty(M,E)$. Note that

$$f \otimes \mu \in C_c^\infty(M, E^* \otimes E \otimes \mathrm{Dens}(M)),$$

that is, $f \otimes \mu(m) = f(m) \otimes \mu(m)$. Note that we have a natural map

$$q : C_c^\infty(M, E^* \otimes E \otimes \mathrm{Dens}(M)) \to C_c^\infty(M, \mathrm{Dens}(M))$$

by pairing E with E^* and a natural map

$$\int : C_c^\infty(M, \mathrm{Dens}(M)) \to \mathbb{C}$$

by integrating over M according to the measure defined by the section of the density bundle. We define

$$\langle i(f), \mu\rangle := \int_M q(f \otimes \mu).$$

Therefore, the definition of generalized sections indeed generalizes smooth sections.

Proposition 7.2.2. *Let X be either a smooth or an F-analytic manifold. Then*

$$\overline{C_c^\infty(X)}^w = C^{-\infty}(X).$$

Proof. Recall that $C^{-\infty}(X) = \mu_c^\infty(X)^*$. Given a topological vector space V, for $W \subseteq V^*$ the space W is dense with respect to the weak topology if and only if $W^\perp = \{v \in V : \langle w, v\rangle = 0 \; \forall w \in W\} = \{0\}$. To see the relevant direction, if $W^\perp = \{0\}$, we will show that for every $\xi \in V^*$, finite set $S \subset V$ and $\epsilon > 0$ we can find $w \in W$ such that $\xi_{|_S} = w_{|_S}$. Given such $\xi \in V^*$, $S = \{v_1, \dots, v_n\}$ and $\epsilon > 0$, assume S is a linearly independent set, and consider $\rho : V^* \to \mathbb{R}^n$ by $\rho(\eta) = (\langle \eta, v_1\rangle, \dots, \langle \eta, v_n\rangle)$. The map $\rho_{|_W}$ is onto, since otherwise there exist $c_i \in \mathbb{R}$ such that $\sum_{i=1}^n c_i\langle w, v_i\rangle = 0$ for all $w \in W$ (it must lie in some hyperplane, and all hyperplanes are of this form), but this means that $\langle w, \sum_{i=1}^n c_i v_i\rangle = 0$ implying $\sum_{i=1}^n c_i v_i \in W^\perp = \{0\}$. The surjectivity of $\rho_{|_W}$ allows us to find the desired $w \in W$.
Thus it is enough to show that given $\eta \in \mu_c^\infty(X)$, if $\langle f, \eta\rangle = 0$ for all $f \in C_c^\infty(X)$ then $\eta = 0$.
Assume X is a smooth manifold. Given a non-zero measure η, there exists some $\mathbb{R}^n \simeq U \subset X$ such that $\eta_{|U} \neq 0$, to see this either use the fact that distributions form a sheaf, or view it as a positive function on Borel sets. Now, since $U \simeq \mathbb{R}^n$ we must have that $\eta_{|_U} = g \cdot \mu_{\text{Haar}}$ where $g \in C^\infty(\mathbb{R}^n)$. Taking some cutoff function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_{|_{B_1(0)}} \equiv 1$ and $\psi \geq 0$ implies the desired result as $\langle g\psi, \eta\rangle = \langle g\psi, g \cdot \mu_{\text{Haar}}\rangle = \langle g^2\psi, \mu_{\text{Haar}}\rangle > 0$ as this is an integral of a positive function.
For an F-analytic manifold we do the same procedure only this time ψ is the indicator function of the open unit ball in F^n. $\square$

Exercise 7.2.3. *Let M, N be either smooth or F-analytic manifolds and let E and I be complex vector bundles over M and N respectively.*

(i) Show that the natural map $C_c^\infty(M, E) \otimes C_c^\infty(N, I) \to C_c^\infty(M \times N, E \boxtimes I)$ is an embedding with dense image, and is an isomorphism if M, N are F-analytic manifolds.

(ii) Show that the natural map

$$\Phi : \text{Dist}(M, E) \otimes \text{Dist}(N, I) \to \text{Dist}(M \times N, E \boxtimes I)$$

given by

$$\langle \Phi(\xi \otimes \eta), F\rangle := \langle \eta, f\rangle, \text{ where } f \text{ is given by } f(y) := \langle \xi, F|_{\mathbb{R}^n \times \{y\}}\rangle$$

is an embedding with dense image.

(iii) Show that the natural map

$$\mathrm{Dist}(M \times N, E \boxtimes I) \to L(C_c^\infty(M,E), \mathrm{Dist}(N,I))$$

is an embedding with dense image, and is an isomorphism if M, N are F-analytic manifolds.

Definition 7.2.4. *Let X be an ℓ-space and $\mathcal{F}$ a sheaf over X. Define $\mathcal{F}_c(X)$ to be the space of* compactly supported global sections *of $\mathcal{F}$, that is all $s \in \mathcal{F}(X)$ such that $s|_{K^C} = 0$ outside some compact K. Define $C_c^\infty(X, \mathcal{F}) := \mathcal{F}_c(X)$ and* $\mathrm{Dist}(X,\mathcal{F}) = C_c^\infty(X,\mathcal{F})^*$.

Theorem 7.2.5. *Let $i : Z \hookrightarrow X$ be ℓ-spaces where Z is closed. Then:*

(1) $\mathrm{Dist}(X,\mathcal{F})|_Z \simeq \mathrm{Dist}(Z, \mathcal{F}|_Z) = i^*(\mathcal{F})$.

(2) We have the following short exact sequence:

$$0 \to \mathrm{Dist}(Z,\mathcal{F}|_Z) \to \mathrm{Dist}(X,\mathcal{F}) \to \mathrm{Dist}(U, \mathcal{F}|_U) \to 0.$$

We now want to prove the following important theorem:

Theorem 7.2.6. *Let $N \subseteq M$ be a closed submanifold of a real manifold M, and let E a bundle over M. Then there is a canonical filtration $F_i \subseteq \mathrm{Dist}(M,E)$ such that:*

(1) Every $\xi \in F_i$ is supported on N.

(2) F_i is locally exhaustive, i.e. $\bigcup_{i=1}^{\infty} F_i$ ***is locally*** $\mathrm{Dist}_N(M,E)$.

(3) $F_i/F_{i-1} \simeq \mathrm{Dist}(N, E|_N \otimes \mathrm{Sym}^i(\mathrm{CN}_N^M))$.

In order to prove the theorem, we would like to define the notion of derivatives of smooth sections $f \in C_c^\infty(M,E)$. Alas, the value of the derivative depends on the chart defined on M, so it is not well defined. Fortunately, the notion of vanishing of derivatives of certain order is well defined as the following exercise shows:

Exercise 7.2.7. *Let $f \in C^\infty(\mathbb{R}^n)$ such that $f^{(\alpha)}(0) = 0$ for every multi-index α with $|\alpha| < k$, and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism such that $\varphi(0) = 0$. Furthermore let $g \in C^\infty(\mathbb{R}^n)$ be a nowhere vanishing function, and set $\widetilde{f}(x) = g(x)(f \circ \varphi^{-1}(x))$.*

(1) Show that $\widetilde{f}^{(\alpha)}(0) = 0$ for every multi-index α with $|\alpha| < k$.

(2) Show that:

$$\left(\frac{\partial^k}{\partial v_1 \ldots \partial v_k}\widetilde{f}\right)(0) = \left(\frac{\partial^k}{\partial\left((d\varphi)v_1\right)\ldots\partial\left((d\varphi)v_k\right)} f\right)(0)g(0).$$

(3) Find a counter example for part (1) if $f^{(i)}(0) \neq 0$ for some $|i| < k$.

Remark 7.2.8. *As a consequence of this exercise, given any $f \in C_c^\infty(M,E)$ whose first $k-1$ derivatives vanish we can define the k*-th differential symbol *of f denoted*

$d_x^k f : T_xM \times \ldots \times T_xM \to E_x$ *by*

$$d_x^k f(\xi_{1,i}, \ldots, \xi_{k,i}) = \left(\frac{\partial^k}{\partial \xi_{1,i} \ldots \partial \xi_{k,i}} (f \circ \varphi_i^{-1}) \right) (0),$$

where φ_i *is a local chart and* $\xi_{1,i} = (\varphi_i \circ \gamma_1)'(0)$ *are tangent vectors. If we choose a different chart* φ_j *we get that*

$$d_x^k f(\xi_{1,j}, \ldots, \xi_{k,j}) = \left(\frac{\partial^k}{\partial \xi_{1,j} \ldots \partial \xi_{k,j}} (f \circ \varphi_j^{-1}) \right) (0) = \left(\frac{\partial^k}{\partial \xi_{1,j} \ldots \partial \xi_{k,j}} (f \circ \varphi_i^{-1} \circ \varphi) \right) (0)$$

where $\varphi := \varphi_i \circ \varphi_j^{-1}$*. By the discussion above, we get that*

$$\left(\frac{\partial^k}{\partial \xi_{1,j} \ldots \partial \xi_{k,j}} (f \circ \varphi_j^{-1}) \right) (0) = \left(\frac{\partial^k}{\partial (d\varphi)\xi_{1,j} \ldots \partial (d\varphi)\xi_{k,j}} (f \circ \varphi_i^{-1}) \right) (0),$$

but as

$$d_x\varphi(\xi_{1,j}) = d_x\varphi \cdot (\varphi_i \circ \gamma_1)'(0) = (\varphi \circ \varphi_i \circ \gamma_1)'(0) = \xi_{1,i}$$

we have that $d_x^k f(\xi_{1,j}, \ldots, \xi_{k,j}) = d_x^k f(\xi_{1,i}, \ldots, \xi_{k,i})$ *so this is well defined.*

Proof of Theorem 7.2.6. Note that we can identify $d_x^k f \in \mathrm{Sym}^k(T_x^*M) \otimes E_x$. Let $N \subseteq M$ be a submanifold. Define:

$$F_N^i(C_c^\infty(M,E)) = \{f \in C_c^\infty(M,E) : \forall x \in N, \text{ the first } k-1 \text{ derivatives of } f \text{ vanish}\}.$$

Choose trivializations $M|_U \simeq \mathbb{R}^n$ and $N|_{U \cap N} \simeq \mathbb{R}^k$. We showed that $F_W^{i-1}(V)/F_W^i(V) \cong C_c^\infty(W, \mathrm{Sym}^i(W^\perp) \otimes E_x)$ using the map $f \mapsto d_x^k f$. Hence we get that:

$$F_N^i/F_N^{i-1} \simeq C_c^\infty(N, E|_N \otimes_{\mathbb{C}} \mathrm{Sym}^i(\mathrm{CN}_N^M)).$$

This gives a canonical filtration $F_i \subseteq \mathrm{Dist}_N(M,E)$ such that

$$F_i/F_{i-1} \simeq \left(F_N^i/F_N^{i-1}\right)^* \simeq C_c^\infty(N, E|_N \otimes_{\mathbb{C}} \mathrm{Sym}^i(\mathrm{CN}_N^M))^* = \mathrm{Dist}(N, E|_N \otimes_{\mathbb{C}} \mathrm{Sym}^i(\mathrm{CN}_N^M)).$$

□

Corollary 7.2.9. *We have*

$$\mathrm{Gr}_i(C^{-\infty}(M,E)_N) = C^{-\infty}(N, E|_N \otimes \mathrm{Dens}(M)^*|_N \otimes \mathrm{Sym}^i(\mathrm{N}_N^M) \otimes \mathrm{Dens}(N)).$$

Proof. We have

$$\begin{aligned} \mathrm{Gr}_i(C^{-\infty}(M,E)_N) &= \mathrm{Gr}_i(\mathrm{Dist}_N(M, E^* \otimes \mathrm{Dens}(M))) \\ &\simeq \mathrm{Dist}(N, E^*|_N \otimes \mathrm{Dens}(M)|_N \otimes \mathrm{Sym}^i(\mathrm{CN}_N^M)) = \\ &= C^{-\infty}(N, E|_N \otimes \mathrm{Dens}(M)^*|_N \otimes \mathrm{Sym}^i(\mathrm{N}_N^M) \otimes \mathrm{Dens}(N)). \end{aligned}$$

□

8. Operations on generalized functions

In this section we assume X and Y are either ℓ-spaces, analytic F-manifolds (with or without complex bundles over them), or smooth manifolds.

Definition 8.0.1. *Let $\varphi : X \to Y$ be a map. We define the* pullback $\varphi^* : C^\infty(Y) \to C^\infty(X)$ *by $\varphi^*(f) = f \circ \varphi$. It is easy to see that if φ is proper then $\varphi^* : C_c^\infty(Y) \to C_c^\infty(X)$. By dualizing, we get an operation $\varphi_* : \mathrm{Dist}(X) \to \mathrm{Dist}(Y)$ on distributions, which we call* pushforward, *by*

$$\varphi_*(\xi)(f) := \xi(\varphi^*(f)) = \xi(f \circ \varphi).$$

Note that if φ is not proper then we can define $\varphi_* : \mathrm{Dist}(X)_{\mathrm{prop}} \to \mathrm{Dist}(Y)$ where $\mathrm{Dist}(X)_{\mathrm{prop}} := \{\xi \in \mathrm{Dist}(X) |\, \varphi|_{\mathrm{supp}(\xi)} \text{ is proper}\}$. We would like to set $\langle \varphi_*\xi, f\rangle = \langle \xi, f \circ \varphi\rangle$, but $f \circ \varphi$ might not be compactly supported. Therefore we choose a cutoff function ρ such that $\rho|_{\mathrm{supp}(\xi)} = 1$ and $\rho|_{U^C} = 0$ where U is a small neighborhood of $\mathrm{supp}(\xi)$ and $\varphi|_{\overline{U}}$ is proper (it is a hard task to find such a function). Hence we can define

$$\langle \varphi_*\xi, f\rangle := \langle \xi, \rho \cdot (f \circ \varphi)\rangle.$$

Note that

$$\mathrm{supp}(\rho \cdot (f \circ \varphi)) \subseteq \mathrm{supp}(\rho) \cap \varphi^{-1}(\mathrm{supp}(f)) \subseteq \varphi|_{\mathrm{supp}(\rho)}^{-1}(\mathrm{supp}(f)).$$

Since $\varphi|_{\mathrm{supp}(\rho)}$ is proper, and f is compactly supported, this is well defined. The definition clearly does not depend on the choice of ρ.
Recall that for vector spaces we had that $\mathrm{Dens}(V) \simeq \mathrm{Haar}(V)$ canonically. Hence we can identify the space of smooth measures $\mu_c^\infty(X)$ with the space of smooth sections of the density bundle $C_c^\infty(X, \mathrm{Dens}(X))$. Note that we can define $\varphi_* : C_c^\infty(X, \mathrm{Dens}(X)) \to \mathrm{Dist}(X)$ by $\langle \varphi_*(\mu), f\rangle = \int_X f d\mu$.

Exercise 8.0.2. *Let X and Y be either smooth or F-analytic manifolds and $\varphi : X \to Y$ a map. Show that the pushforward of a compactly supported distribution is compactly supported, that is $\varphi_*(\mathrm{Dist}_c(X)) \subseteq \mathrm{Dist}_c(Y)$.*

Proposition 8.0.3. *Let X and Y be either smooth or F-analytic manifolds and $\varphi : X \to Y$ be a submersion. Then:*

(1) $\varphi_*(\mu_c^\infty(X)) \subseteq \mu_c^\infty(Y)$.
(2) $\varphi_*(f \cdot |\omega_X|) = g \cdot |\omega_Y|$, where $|\omega_X|$ and $|\omega_Y|$ are non-vanishing densities on X and Y respectively and

$$g(y) = \int_{\varphi^{-1}(y)} f \frac{|\omega_X|}{|\varphi^*\omega_Y|}$$

where $\frac{|\omega_X|}{|\varphi^*\omega_Y|} \otimes |\omega_Y|$ is the image of $|\omega_X|$ under the natural isomorphism

$$\mathrm{Dens}(X)_x \simeq \mathrm{Dens}(\varphi^{-1}(y))_x \otimes \mathrm{Dens}(Y)_{\varphi(x)}.$$

Proof.

(1) We prove the first statement in two steps. Case 1: $X = F^n, Y = F^m$ where $n \geq m$ and $\varphi : F^n \to F^m$ is the natural projection $\varphi(x_1, \ldots, x_n) = x_1, \ldots, x_m$. Recall that $\mathrm{Haar}(X) \simeq \mathrm{Haar}(Y) \otimes \mathrm{Haar}(X/Y)$ or equivalently

that

$$\mathrm{Dens}(X) \simeq \mathrm{Dens}(Y) \otimes \mathrm{Dens}(X/Y).$$

Let $\phi \in C_c^\infty(X, \mathrm{Dens}(X))$ and note that $\phi = f \cdot d\mu_X$ where $f \in C_c^\infty(X)$ and μ_X is a normalized Haar measure (taking the value 1 on the unit ball of $X = F^n$), so we can write $\mu_X = \mu_Y \otimes \mu_{X/Y}$. By definition, for any $g \in C_c^\infty(Y)$ we have:

$$\langle \varphi_*(\phi), g\rangle = \langle \phi, g \circ \varphi\rangle = \int_X f \cdot (g \circ \varphi) d\mu_X = \int_Y \int_{X/Y} f \cdot (g \circ \varphi) d\mu_Y \otimes \mu_{X/Y}.$$

It is compactly supported, and since $g \circ \varphi(x_1, \ldots, x_n) = g(x_1, \ldots, x_m)$ depends only on Y we have

$$\langle \varphi_*(\phi), g\rangle = \int_Y \left(\int_{X/Y} f \cdot d\mu_{X/Y} \right) \cdot g d\mu_Y = \int_Y \widetilde{f} \cdot g d\mu_Y$$

where $\widetilde{f} \in C_c^\infty(Y)$. Hence $\varphi_*(\phi)$ is a smooth measure.

Case 2 - general case: Let $\varphi : X \to Y$ be a submersion. Take trivializing covers $Y = \bigcup_{j\in J} V_j$ and $X = \bigcup_{i \in I} U_i$ such that $\varphi(U_i) \subseteq V_j$. For any i, j such that $\varphi(U_i) \subseteq V_j$ we can choose isomorphisms $\tau_i : U_i \simeq F^n$ and $\psi_j : V_j \simeq F^m$ (if X and Y are F-analytic we choose isomorphisms to some powers of $\mathcal{O}_F$) such that $\psi_j \circ \varphi \circ \tau_i^{-1}$ is the natural projection $F^n \to F^m$ (respectively $\mathcal{O}_F^n \to \mathcal{O}_F^m$ for F-analytic). Hence $\left(\psi_j \circ \varphi \circ \tau_i^{-1}\right)_* (\mu_c^\infty(F^n)) \subseteq (\mu_c^\infty(F^m)$ and $\varphi_*(C_c^\infty(U_i, \mathrm{Dens}(U_i))) \subseteq C_c^\infty(V_j, \mathrm{Dens}(V_j))$.

Now, let $\phi \in C_c^\infty(X, \mathrm{Dens}(X))$. Using partition of unity, we can write $\phi = \sum_{i\in I} f_i \mu_i$ where $f_i\mu_i \in C_c^\infty(U_i, \mathrm{Dens}(U_i))$. Note that this is a finite sum since ϕ is compactly supported and observe that:

$$\varphi_*(\phi) = \varphi_* \left(\sum_{i\in I} f_i \mu_i \right) = \sum_{i \in I} \varphi_*(f_i\mu_i) = \sum_{i\in I} g_i \mu_i'$$

where $g_i \in C_c^\infty(V_j, \mathrm{Dens}(V_j))$. Each $g_i\mu_i'$ is a smooth compactly supported measure, so the sum $\sum_{i\in I} g_i \mu_i'$ is a smooth section of the density bundle and we are done.

(2) Since φ is a submersion for any $\varphi(x) = y \in Y$ the fiber $\varphi^{-1}(y)$ is a submanifold of X and the following sequence is exact:

$$0 \to T_x \varphi^{-1}(y) \to T_x(X) \to T_{\varphi(x)}(Y) \to 0.$$

Since this is an exact sequence of vector spaces it splits so $T_x(X) = T_x\varphi^{-1}(y) \oplus T_{\varphi(x)}(Y)$ and by dualizing we get that

$$T_x^*(X) = T_x^* \varphi^{-1}(y) \oplus T_{\varphi(x)}^*(Y).$$

This implies that $\mathrm{Dens}(X)_x = \mathrm{Dens}(\varphi^{-1}(y))_x \otimes (\mathrm{Dens}(Y))_{\varphi(x)}$.

We now reduce the problem to a small neighborhood. As before take trivializing covers $Y = \bigcup_{j\in J} V_j$ and $X = \bigcup_{i\in I} U_i$ such that $\varphi(U_i) \subseteq V_j$, and choose appropriate isomorphisms τ_i and ψ_j for $\varphi(U_i) \subseteq V_j$ such that $\psi_j \circ \varphi \circ \tau_i^{-1}$ is the natural projection $F^n \to F^m$ (resp. $\mathcal{O}_F^n \to \mathcal{O}_F^m$).

We need to prove that for every $h \in C_c^\infty(Y)$ we have

$$\langle \varphi_*(f\,|\omega_X|), h\rangle = \langle f\,|\omega_X|\,, h\circ\varphi\rangle = \langle g\cdot|\omega_Y|\,, h\rangle,$$

where g is as in the statement of the proposition. Construct a partition of unity $f = \sum_{i\in I} f_i$ with respect to $\{U_i\}_{i\in I}$. Then it is enough to prove the claim for $f_i\,|\omega_X|$ as then:

$$\varphi_*(f\,|\omega_X|)(h) = \varphi_*(\sum f_i\,|\omega_X|)(h) = \sum \int_Y g_i h\,|\omega_Y|$$

where $g_i(y) = \int_{\varphi^{-1}(y)} f_i\eta$ since $\mathrm{supp}(f_i\eta) \subset U_i$. As $g = \sum_{i\in I} g_i$ we would have that $g(y) = \int_{\varphi^{-1}(y)} f\eta$ as required.

In (1) we showed the case where φ is a projection. Using the fact that for diffeomorphisms pushforward and pullback are inverse to one another and using part (1) we get that:

$$\begin{aligned}\psi_j\circ\varphi_*(f_i\,|\omega_X|) &= \psi_j\circ\varphi\circ(\tau_i^{-1})_*((\tau_i^{-1})^*\,(f_i\,|\omega_X|))\\ &= \psi_j\circ\varphi\circ(\tau_i^{-1})_*(f_i\circ\tau_i^{-1}\cdot\left|(\tau_i^{-1})^*\omega_X\right|) = \widetilde{g_i}\left|(\psi_j^{-1})^*\omega_Y\right|\end{aligned}$$

where $\widetilde{g_i}(x) = \int_{\tau_i\circ\varphi^{-1}\circ\psi_j^{-1}(x)} f_i\circ\tau_i^{-1}\left|\frac{(\tau_i^{-1})^*\omega_X}{(\varphi\circ\tau_i^{-1})^*\omega_Y}\right|$. Set $g_i := \widetilde{g_i}\circ\psi_j$, then $\widetilde{g_i} = g_i\circ\psi_j^{-1}$. Hence $\varphi_*(f_i\,|\omega_X|) = g_i\,|\omega_Y|$ where,

$$g_i(y) = \widetilde{g_i}(\psi_j(y)) = \int_{\tau_i\circ\varphi^{-1}(y)} f_i\circ\tau_i^{-1}\left|\frac{(\tau_i^{-1})^*\omega_X}{(\varphi\circ\tau_i^{-1})^*\omega_Y}\right| = \int_{\varphi^{-1}(y)} f_i\left|\frac{\omega_X}{\varphi^*\omega_Y}\right|.$$

□

Definition 8.0.4. *By the proposition, the map $\varphi_* : C_c^\infty(X, \mathrm{Dens}(X)) \to C_c^\infty(Y, \mathrm{Dens}(Y))$ gives rise to a pullback $\varphi^* : C^{-\infty}(Y) \to C^{-\infty}(X)$.*

Exercise 8.0.5. *Let $\varphi : X \to Y$ be a submersion. We can define a pullback $\varphi^* : C^\infty(Y) \to C^\infty(X)$ both by $\varphi^*(f) = f\circ\varphi$ and by the restriction of the map $\varphi^* : C^{-\infty}(Y) \to C^{-\infty}(X)$ to the subspace $C^\infty(Y) \subset C^{-\infty}(Y)$. Show that these two definitions coincide.*

We can generalize the pushforward and pullback operations on functions and on distributions to functions and distributions with values in a vector bundle:

Definition 8.0.6. *Let $\varphi : X \to Y$ and let be $\pi : E \to Y$ a bundle.*

(1) Define the pullback of the bundle (E, π) to X by

$$\varphi^*(E) := \{(x, e) \in X \times E : \varphi(x) = \pi(e)\}$$

with the natural projection to X.

(2) *Using (1) define the the pullback of sections*

$$\varphi^*: C^\infty(Y,E) \to C^\infty(X,\varphi^*(E))$$

and by dualizing the pushforward of distributions

$$\varphi_*: \mathrm{Dist}(X,\varphi^*(E))_{\mathrm{prop}} \to \mathrm{Dist}(Y,E).$$

(3) *Define $\varphi^!(E) := \varphi^*(E) \otimes \varphi^*(\mathrm{Dens}(Y)^*) \otimes \mathrm{Dens}(X)$.*

Proposition 8.0.7. *Let $\varphi: X \to Y$ be a submersion. Then*

$$\varphi_*\left(C_c^\infty(X,\varphi^*(E)\otimes \mathrm{Dens}(X))\right) \subseteq C^\infty(Y, E\otimes \mathrm{Dens}(Y)).$$

In particular, this implies that $\varphi_*\left(C_c^\infty(X,\varphi^!(E))\right) \subseteq C^\infty(Y,E)$.

Proof. As in the proof of the last proposition, we may reduce to the case where $\varphi: X\to Y$ is the natural projection, and $X = F^n$, $Y = F^m$, and $E \simeq F^m \times F^k$ is trivial (resp $\mathcal{O}_F^n$,$\mathcal{O}_F^n$ and $\mathcal{O}_F^m \times \mathcal{O}_F^k$ for F-analytic manifolds). As a consequence, $\varphi^*(E) = F^n \times F^k$ (resp. $\mathcal{O}_F^n \times \mathcal{O}_F^k$). Note reducing is possible since the notion of smoothness of a distribution (that is, it is a smooth measure) is local.
Let $\phi = f\mu \in C_c^\infty(X,\varphi^*(E)\otimes \mathrm{Dens}(X))$. Then we have for any $g \in C^\infty(Y,E)$,

$$\begin{aligned}\langle \varphi_*(\phi), g\rangle = \langle \phi, g\circ\varphi\rangle = \int_X f\cdot(g\circ\varphi)\mu_X = \int_Y \left(\int_{X/Y} f\cdot \mu_{X/Y}\right)\cdot g\mu_Y \\ = \int_Y \widetilde{f}\cdot g\mu_Y = \langle \tilde{f}\mu_Y, g\rangle,\end{aligned}$$

where $\tilde{f} = \int_{X/Y} f\cdot\mu_{X/Y}$ which is smooth, so $\varphi_*(\phi)$ is smooth. $\square$

9. Fourier transform

Definition 9.0.1. *Let G be a locally compact Hausdorff abelian group. Define its Pontryagin dual by,*

$$G^\vee = \{\chi: G\to \mathrm{U}_1(\mathbb{C}) = S^1 \subseteq \mathbb{C} |\, \chi(g_1g_2) = \chi(g_1)\chi(g_2),\ \chi \text{ is cts}\}.$$

The topology on $G^\vee$ is the compact open topology, i.e. a sub-basis of the topology is comprised of sets $M(K,V) = \{\chi\in G^\vee : \chi(K)\subseteq V\}$ where $K\subseteq G$ is compact and $V \subseteq S^1$ is open.

Theorem 9.0.2. *Let G be a locally compact, Hausdorff abelian group, then $G^\vee$ is a locally compact Hausdorff abelian group.*

Proof. We see that characters form an abelian group. Since S^1 is a topological group, the compact open topology on $G^\vee$ is equivalent to the topology of uniform convergence on compact sets. Thus, in order to show that the multiplication and inverse operations are continuous, it is enough to show that if $f_n \to f$ and $g_n \to g$

uniformly on compact sets then $f_n \cdot g_n^{-1} \to f \cdot g^{-1}$ uniformly on compact sets. Now, if $K \subset G$ is compact, note that this follows from the following bound ($\forall x \in K$):

$$|f_n g_n^{-1} - f g^{-1}| \leq |f_n(g_n^{-1} - g^{-1})| + |(f_n - f)g^{-1}| = |g_n - g| + |f_n - f|.$$

Now to show it is locally compact, consider the space $(S^1)^G$ of all functions $f : G \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ with the product topology (i.e. a basis is given by open sets in only finitely many components). It is a compact space by Tychonoff's theorem, and it has the space

$$\tilde{G} = \bigcap_{g_1, g_2 \in G} \{\chi : G \to S^1 : \chi(g_1 g_2) = \chi(g_1)(g_2)\},$$

as a closed subspace, implying that $\tilde{G}$ is compact. Furthermore, for every $S \subseteq G$ and $\epsilon > 0$ the set $A(S, \epsilon) = \{\chi \in (S^1)^G : \chi(S) \subseteq [-\epsilon, \epsilon]\}$ is also closed and compact in $(S^1)^G$ as the complement is a union of sets of the form

$$\{\chi : G \to S^1 : \exists s \in S \text{ s.t. } \chi(s) \in [-\epsilon, \epsilon]^c\}$$

which are open.

In particular, taking an open neighborhood $e \in U \subset G$, the sets $V(U, \epsilon) = A(U, \epsilon) \cap \tilde{G}$ are closed and compact in $(S^1)^G$. Take $0 < \epsilon < \frac{1}{2}$, we show that we have that $V(U, \epsilon) \subseteq G^\vee$. Start with an open $e \in U_0 = U \subset G$, and choose a sequence of neighborhoods $(U_n)_{n=1}^\infty$ in G such that $U_{n+1} \cdot U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$ and set $\epsilon_n = \frac{\epsilon}{2^n}$. Taking $\chi \in V(U_n, \epsilon_n)$, we see that since for $x \in U_{n+1}$ we have that $\chi(x) \in [-\epsilon_n, \epsilon_n]$ and $x^2 \in U_n$ we get $\chi(x^2) = \chi(x)^2 \in [-\epsilon_n^2, \epsilon_n^2] \subseteq [-\frac{\epsilon_n}{2}, \frac{\epsilon_n}{2}]$, implying that $V(U_n, \epsilon_n) \subseteq V(U_{n+1}, \epsilon_{n+1})$.

Now, take $\chi \in V(U, \epsilon)$ and a basic open set $(-\delta, \delta) \subset S^1$ for $\delta > 0$. We have that $[-\epsilon_n, \epsilon_n] \subseteq (-\delta, \delta)$ for n big enough, implying that $e \in U_n \subseteq \chi^{-1}((-\delta, \delta))$ which means that χ is continuous at e. Since χ is a homomorphism, we can show it is continuous everywhere; if $\chi(g) \in W \subset S^1$ and W is open, we have that $(-\delta, \delta) \subseteq \chi(g^{-1})W$ for some $\delta > 0$ and that,

$$\begin{aligned}\chi^{-1}(\chi(g^{-1})W) &= \{y \in G : \chi(y) \in \chi^{-1}(g)W\} = \{y \in G : \chi(gy) \in W\} \\ &= g^{-1}\{gy \in G : \chi(gy) \in W\} = g^{-1}\chi^{-1}(W).\end{aligned}$$

Now, for some $m \in \mathbb{N}_0$ big enough, the following implies that $g \in gU_m \subseteq \chi^{-1}(W)$:

$$U_m \subset \chi^{-1}(\chi(g^{-1})W) = g^{-1}\chi^{-1}(W).$$

We know that $V(U, \epsilon)$ is compact in the product topology, and want to show it is compact with respect to the compact open topology. For this, it is enough to show that any net in $V(U, \epsilon)$ has a converging subnet in the compact open topology. Assume we are given some net $(x_\alpha) \in V(U, \epsilon)$, then it has a subnet $(f_\beta) \to f$ converging in the product topology with $f_\beta, f \in V(U, \epsilon)$. Now, note that $V(U, \epsilon)$ is uniformly equicontinuous, that is if $g_1, g_2 \in G$ and $g_1 g_2^{-1} \in U_n$ then for any

$\chi \in V(U, \epsilon)$,

$$|\chi(g_1) - \chi(g_2)| = |\chi(g_1)\chi^{-1}(g_2) - 1| = |\chi(g_1 g_2^{-1}) - 1| \leq \epsilon_n.$$

Given a basic open neighborhood of the identity character $1_G \in M(K, B_{\epsilon'}(0))$, where K is compact, for every $g \in K$ we have that $g \in U_n g$ (for n big enough). Now, taking any $g' \in U_n g$, we get that $g' g^{-1} \in U_n$ implying that for some big enough β we have that $|f(g) - f_\beta(g)| < \epsilon_n$ and that,

$$|f(g') - f_\beta(g')| \leq |f(g') - f(g)| + |f_\beta(g') - f_\beta(g)| + |f(g) - f_\beta(g)| < 3\epsilon_n.$$

Taking $n > n_g$ such that $\epsilon_n < \frac{\epsilon'}{3}$, we see that $f_\beta \to f$ uniformly on $U_n g$, but since K is compact we can cover it with finitely many sets of the form $U_{n_g} g$, and take $n = \max\limits_{0 \leq i \leq k} \{n_{g_i}\}$ and appropriate β.
To finish off the argument, note that by local compactness every $g \in G$ has a neighborhood $g \in U$ with compact closure $U \subset K$, and we have that $M(K, B_\epsilon(0)) \subseteq V(U, \epsilon)$ for an appropriate $\frac{1}{2} > \epsilon > 0$. $\square$

Exercise 9.0.3. *Let G be a locally compact, Hausdorff abelian group. Show that if G is compact then $G^\vee$ is discrete, and that if G is discrete then $G^\vee$ is compact.*

Theorem 9.0.4. *For a locally compact abelian group G, we have that the natural map $\varphi : G \to G^{\vee\vee}$ defined by $g \mapsto \varphi_g$, where $\varphi_g(\chi) = \chi(g)$, is an isomorphism $G^{\vee\vee} \simeq G$.*

Proof. This is complicated. Need a reference. $\square$

Exercise 9.0.5. *Let G be a locally compact, Hausdorff abelian group, and $H \leq G$ a closed subgroup. Show that:*

(1) *Pontryagin duality is a contravariant endofunctor in the category of locally compact abelian groups.*
(2) *$H^\vee \simeq G^\vee / H^\perp$ where $H^\perp = \{\chi \in G^\vee : \chi(h) = 1 \ \forall h \in H\}$, and that if H and G are vector spaces then this is a homeomorphism (Hint: use an appropriate version of the Hahn-Banach theorem).*

Example 9.0.6.

(1) *For any finite abelian group G we have that $G \simeq G^\vee$.*
(2) *The dual of $\mathrm{U}_1(\mathbb{C}) = S^1$ is $\mathbb{Z}$.*
(3) *We have that $\mathbb{R}^\vee \simeq \mathbb{R}$.*

Exercise 9.0.7. *Let V be a topological vector space over a local field F. Then $V^* \otimes_F F^\vee \simeq V^\vee$.*

Definition 9.0.8. *Let G be a locally compact Hausdorff abelian group. The map $\mathcal{F} : \mu_c(G) \to C(G^\vee)$ defined by $\mathcal{F}(\mu)(\chi) = \int \chi d\mu$ is called the* Fourier transform.

Exercise 9.0.9. *Show the following:*

(1) $\mathcal{F}$ is continuous.
(2) Let G be a locally compact abelian group. For a character $\tau : G \to S^1$ define $sh_h(\tau)(x) = \tau(x+h)$. Show that for $\eta \in \mu_c^\infty(G)$ and $g \in G$:
(a) $\mathcal{F}(sh_g(\eta))(\chi) = \chi(g)\mathcal{F}(\eta)(\chi)$ for all $\chi \in G^\vee$.
(b) $\mathcal{F}(\chi\eta) = sh_{\chi^{-1}}(\mathcal{F}(\eta))$ for all $\chi \in G^\vee$.

Definition 9.0.10. *Let X_1 and X_2 be locally compact topological vector spaces and let $\mu_1 \in \mu_c^\infty(X_1)$ and $\mu_2 \in \mu_c^\infty(X_2)$. We define the external tensor product of such measures $\mu_1 \boxtimes \mu_2 \in \mu_c^\infty(X_1 \times X_2)$. In addition, If $X_1 = X_2 = G$, then we define the convolution of these measures by $\mu_1 * \mu_2 := m_*(\mu_1 \boxtimes \mu_2)$ where $m : G \times G \to G$ is the multiplication map.*

Fact 9.0.11. *For two measures $\alpha, \beta \in \mu_c^\infty(G)$ we have that $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$.*

Definition 9.0.12. *Let V be a finite dimensional vector space over a local field F. Define the space of* Schwartz functions *$\mathcal{S}(V)$ on V by:*
(1) If F is non-archimedean, then $\mathcal{S}(V) = C_c^\infty(V)$, i.e. locally constant functions on V.
(2) If F is archimedean, then
$$\mathcal{S}(V) = \{f \in C^\infty(V) | \forall i \in \mathbb{N}^n, p \in F[V],\ sup\left|\partial^i f \cdot p(x)\right| < \infty\}.$$
In other words it is the space of rapidly decreasing smooth functions on V.

Proposition 9.0.13. *The Fourier transform $\mathcal{F} : S(V, \mathrm{Haar}(V)) \to S(V^\vee)$ is continuous for an Archimedean V and its image is indeed contained in $S(V^\vee)$ in both cases.*

Proof. Assume V is a real vector space of dimension n, and recall that the topology on $S(V)$ is determined by the semi-norms $\|f\|_{\alpha,\beta} = \sup\limits_{x \in V} |\Phi_\alpha(x) \frac{\partial^\beta f(x)}{\partial x^\beta}|$ where $\alpha, \beta \in \mathbb{N}_0^n$ and $\Phi_\alpha(x) = \prod\limits_{j=1}^n x_j^{\alpha_j}$. It is enough to show that for every $f \in C_c^\infty(V, \mathrm{Haar}(V))$ and semi-norm $\|\cdot\|_{\alpha,\beta}$ on $S(V^\vee)$ there exists a semi-norm $\|\cdot\|'$ on $S(V, \mathrm{Haar}(V))$ and a positive constant C such that $\|\mathcal{F}(f)\|_{\alpha,\beta} \le C\|f\|'$. Now, recall that,
$$\frac{i\partial\mathcal{F}(f)}{\partial \xi_j} = \int\limits_{\mathbb{R}^n} \frac{i\partial}{\partial \xi_j}(e^{-i\xi\cdot x} f(x))dx = \mathcal{F}(x_j f),$$
where one can differentiate directly using the definition to verify the above procedure. The other side of the coin is given by integration by parts,
$$\xi_j \mathcal{F}(f) = \int\limits_{\mathbb{R}^n} \xi_j e^{-i\xi\cdot x} f(x)dx = \left[-e^{-i\xi\cdot x} f(x)\right]_{-\infty}^{\infty} - \int\limits_{\mathbb{R}^n} \frac{\xi_j}{-i\xi_j} e^{-i\xi\cdot x} \frac{\partial f(x)}{\partial x_j} dx = \mathcal{F}(\frac{-i\partial(f)}{\partial x_j}).$$
Note that since the functions $e^{-i\xi\cdot x}$ converge weakly to zero as distributions as $|\xi| \to \infty$ we get that Schwartz measures are mapped into $S(V^\vee)$. We can now

bound $\mathcal{F}(f)$ properly using the above relations:

$$\|\mathcal{F}(f)\|_{\alpha,\beta} = \sup_{x^\vee \in V^\vee} \left|\Phi_\alpha(x^\vee)\frac{(-i\partial)^\beta \mathcal{F}(f)(x^\vee)}{\partial(x^\vee)^\beta}\right| = \sup_{x^\vee \in V^\vee} \left|\int\limits_V x^\vee \frac{(-i\partial)^\alpha(\Phi_\beta(-x)f)}{\partial x^\alpha}\mu\right|$$
$$\leq \sup_{x^\vee \in V^\vee} \int\limits_V \left|x^\vee \frac{\partial^\alpha(\Phi_\beta(-x)f)}{\partial x^\alpha}\right|\mu \leq C \sup_{x\in V}\left((1+|x|)^{n+1}\left|\frac{\partial^\alpha(\Phi_\beta(|x|)f)}{\partial x^\alpha}\right|\right).$$

where $C = \int\limits_V \frac{1}{(1+|x|)^{n+1}} d\mu(x)$. Since the last expression is a linear combination of norms of the form $\|f\|_{\alpha',\beta'}$ for $|\alpha'| \leq |\alpha| + n + 1$ and $|\beta'| \leq |\beta|$, this implies that $\mathcal{F}$ is continuous. Note that we can also use this to show that $\mathcal{F}(f)$ is Schwartz, since if all the norms $\|\cdot\|_{\alpha,\beta}$ are bounded then the value of $|\Phi_\alpha(x)\frac{\partial^\beta f(x)}{\partial x^\beta}|$ decays to 0 as $|x| \to \infty$ for every α and β.
The proof for vector spaces over non-Archimedean fields is analogous. □

Definition 9.0.14. *Let $S(V)$ be the space of Schwartz functions on V.*

1. *We call $\xi \in S^*(V)$ the space of* tempered distributions *and $\mathcal{G}(V) := S^*(V, \mathrm{Haar}(V))$ the space of* tempered generalized functions.
2. *Finally, we define the Fourier transform on tempered distributions via duality:*
$$\mathcal{F}^* : S^*(V^\vee) \to \mathcal{G}(V) := S^*(V, \mathrm{Haar}(V)).$$
Taking $V := V^\vee$ we get $\mathcal{F}^* : S^*(V) \to \mathcal{G}(V^\vee)$.

Theorem 9.0.15. *The definition of Fourier transform of distributions is consistent with the definition given for functions. In other words $\mathcal{F}^*|_{\mathcal{S}(V,\mathrm{Haar}(V))} = \mathcal{F}$.*

Proof. Let $f(x)\cdot dx \in \mathcal{S}(V, \mathrm{Haar}(V))$ and $g(\chi)\cdot d\chi \in \mathcal{S}(V^\vee, \mathrm{Haar}(V^\vee))$. Then by definition,

$$\langle \mathcal{F}^*(f(x)\cdot dx), g(\chi)\cdot d\chi\rangle := \langle f(x)\cdot dx, \mathcal{F}(g(\chi)d\chi)\rangle = \int_V f(x)\mathcal{F}(g(\chi)\cdot d\chi)(x)dx$$

where $\mathcal{F}(g(\chi)\cdot d\chi)(x) := \int_{V^\vee} \chi(x)g(\chi)d\chi$. Therefore we have:

$$\int_V f(x)\mathcal{F}(g(\chi)\cdot d\chi)(x)dx = \int_V f(x)\int_{V^\vee}\chi(x)g(\chi)d\chi dx = \int_{V^\vee}\left(\int_V \chi(x)f(x)dx\right)g(\chi)d\chi$$
$$= \int_{V^\vee}\left(\mathcal{F}(f)(\chi)\right)g(\chi)d\chi = \langle \mathcal{F}(f(x)\cdot dx), g(\chi)\cdot d\chi\rangle.$$

□

Remark 9.0.16. *We will usually omit the $*$ from the $\mathcal{F}^*$ notations, this should cause no confusion.*

In the following argument we would like to show the Fourier transform is a unitary operator. For this we will first need to define a pairing between $\mathrm{Haar}(V)$ and $\mathrm{Haar}(V^\vee)$. Given $\alpha \in \mathrm{Haar}(V)$ and $\beta \in \mathrm{Haar}(V^\vee)$ we can define such a pairing as follows. We choose $f \in C_c^\infty(V^\vee)$ such that $f(0) = 1$ and then define $\langle \alpha, \beta\rangle := \langle \mathcal{F}(\alpha), f \cdot \beta\rangle$.

Exercise 9.0.17.

(1) *Show this is well defined. That is, given a different* $g \in C_c^\infty(V^\vee)$ *such that* $g(0) = 1$, *show that* $\langle \mathcal{F}(\alpha), (f - g) \cdot \beta\rangle = 0$.
(2) *Show that* $\mathrm{Haar}(V^\vee) \simeq_{\mathrm{can}} \mathrm{Haar}(V)^*$.

Exercise 9.0.18. *Show that* $\mathcal{F}(\delta_a)(t) = \exp(at)$.

Definition 9.0.19. *We define a map* $\mathcal{F}_n : S^*(V, \mathrm{Haar}(V)^{\otimes n}) \to S^*(V^\vee, \mathrm{Haar}(V^\vee)^{\otimes(1-n)})$ *such that* $\mathcal{F}_0$ *is the Fourier transform. We use the following isomorphisms:*

(1) *The pairing* $\mathrm{Haar}(V^\vee) \simeq_{\mathrm{can}} \mathrm{Haar}(V)^*$.
(2) *The identification* $S^*(V, \mathrm{Haar}(V)^{\otimes n}) \simeq S^*(V) \otimes \mathrm{Haar}(V)^{\otimes -n}$.
(3) *The identification* $S^*(V^\vee, \mathrm{Haar}(V^\vee)^{\otimes(1-n)}) \simeq S^*(V^\vee, \mathrm{Haar}(V^\vee)) \otimes \mathrm{Haar}(V^\vee)^{\otimes n}$.

The first item was shown in the previous exercise. The second identification is as follows. Given $\xi \otimes \beta \in S^*(V) \otimes \mathrm{Haar}(V^\vee)^{\otimes n}$ *and* $f \cdot \alpha \in S(V, \mathrm{Haar}(V)^{\otimes n})$ *we have that* $\langle \xi \otimes \beta, f\alpha\rangle = \langle \xi, f\rangle\langle \beta, \alpha\rangle$. *The third identification is similar.*
Finally, we define the map by applying the Fourier transform on the first coordinate of the right hand side of (2) *and by applying the canonical map* (1) *on the second coordinate.*

Proposition 9.0.20. *We have that* $\mathcal{F}_1 \circ \mathcal{F}_0 = \mathrm{flip}$ *where* $\langle \mathrm{flip}(\xi), f(x)\mu\rangle = \langle \xi, f(-x)\mu\rangle$.

Proof. Note that $\mathrm{span}\{\delta_x\}_{x \in V}$ is a dense subspace of $S^*(V)$ with respect to the weak topology. Hence it is enough to show that $\mathcal{F}_1 \circ \mathcal{F}_0(\delta_a) = \delta_{-a}$ for all $a \in V$.
Note that $\langle \mathcal{F}_0(\delta_0), f\beta\rangle := \langle \delta_0, \mathcal{F}_0(f\beta)\rangle = \int_{V^\vee} f d\beta$, this implies $\mathcal{F}_0(\delta_0) = 1$.
As before, $\mathcal{F}_1 : S^*(V^\vee, \mathrm{Haar}(V^\vee)) \to S^*(V)$ is defined by identifying : $S^*(V^\vee, \mathrm{Haar}(V^\vee))$ with $S^*(V^\vee) \otimes \mathrm{Haar}(V)$. Under this identification, $1 \cdot \mu$ for a choice of a Haar measure μ on $V^\vee$ is identified with $1 \otimes \eta$ where $\eta \in \mathrm{Haar}(V)$ and $\langle \mu, \eta\rangle = 1$.
Given $f \in S(V)$, we have that (note we are using Theorem 9.0.15):

$$\langle \mathcal{F}_1(1 \cdot \mu), f\rangle = \langle \mathcal{F}^*(1) \otimes \eta, f \cdot \eta \otimes \mu\rangle = \langle \mathcal{F}(1), f \cdot \eta\rangle\langle \eta, \mu\rangle = \langle \delta_0, f \cdot \eta\rangle \cdot 1 = f(0),$$

so $\mathcal{F}_1 \circ \mathcal{F}_0(\delta_0) = \delta_0$.
Using Exercise 9.0.9, we now see that (here $\chi(a)$ is the function which substitutes the value a in a given character χ):

$$\mathcal{F}_1 \circ \mathcal{F}_0(\delta_a) = \mathcal{F}_1 \circ \mathcal{F}_0(\mathrm{sh}_a(\delta_0)) = \mathcal{F}_1(\chi(a)\mathcal{F}_0(\delta_0)) = \mathrm{sh}_{-a}\mathcal{F}_1 \circ \mathcal{F}_0(\delta_0) = \delta_{-a}.$$

By continuity of $\mathcal{F}_0$ and $\mathcal{F}_1$ this implies that $\mathcal{F}_1 \circ \mathcal{F}_0 = \mathrm{flip}$. $\square$

Definition 9.0.21. *Let* F *and* K *be local fields and* $\chi : F^\times \to K^\times$ *a character. For a* 1*-dimensional space* V *over* F *we define a functor by:*

$$\chi(V) := \{\varphi : V^* \to K : \varphi(\alpha v) = \chi(\alpha)\varphi(v) \forall \alpha \in F^\times, v \in V^*\}.$$

Example 9.0.22. *Let $\chi : F^\times \to F^\times$ be the character $x \mapsto x^2$ and let V be a one dimensional vector space of F. Then*

$$\chi(V) := \{\varphi : V^* \to K : \varphi(\alpha f) = \alpha^2 \varphi(f)\}.$$

Note that $\chi(V) \simeq_{\text{can}} V \otimes V$ by $v \otimes w \mapsto \varphi_v \cdot \varphi_w$. Indeed, given $\psi \in V^*$, we have $\varphi_v \cdot \varphi_w(\psi) = \psi(v) \cdot \psi(w)$ and $\varphi_v \cdot \varphi_w(a\psi) = a\psi(v) \cdot a\psi(w) = a^2 \varphi_v \cdot \varphi_w(\psi)$.

Definition 9.0.23. *Let V be a 1-dimensional vector space over $\mathbb{R}$.*

(1) A positive structure *on V is a non trivial subset $P \subseteq V$ such that $\mathbb{R}_{\geq 0} \cdot P = P$.*
(2) If V has a positive structure, we define
$$V^\alpha := |V|^\alpha = \{\varphi : V^* \to \mathbb{R} : \varphi(\beta f) = |\beta|^\alpha \cdot \varphi(f)\}.$$

Exercise 9.0.24. *Let V be a real 1-dimensional vector space with a positive structure.*

(1) Show that:
   (a) $V \simeq_{\text{can}} |V|$.
   (b) $V^{\alpha+\beta} \simeq_{\text{can}} V^\alpha \otimes V^\beta$ where $\alpha, \beta \in \mathbb{Q}^\times$.
(2) Deduce that $\mathrm{Haar}(V)^\alpha \otimes \mathrm{Haar}(V)^\beta \simeq \mathrm{Haar}(V)^{\alpha+\beta}$.

Definition 9.0.25. *For $\alpha \in \mathbb{Q}$ we define similarly to the procedure defined above,*

$$\mathcal{F}_\alpha : S^*(V, \mathrm{Haar}(V)) \to S^*(V^\vee, \mathrm{Haar}(V^\vee)^{1-\alpha}).$$

In particular, choosing $\alpha = \frac{1}{2}$ we have:

$$\mathcal{F}_{\frac{1}{2}} : S^*(V, \mathrm{Haar}(V)^{\frac{1}{2}}) \to S^*(V^\vee, \mathrm{Haar}(V^\vee)^{\frac{1}{2}}).$$

Theorem 9.0.26 (Functoriality of Fourier transform)**.** *Let $W \subset V$ be vector spaces over a local field, denote the inclusion of W in V by i, and set $p : V^\vee \to W^\vee$ for the induced linear map on the duals, then the following diagrams commute:*

$$\begin{tikzcd} S(V) \arrow[r, "i^*"] \arrow[d, "\mathcal{F}"'] & S(W) \arrow[d, "\mathcal{F}"'] \\ S(V^\vee, \mathrm{Haar}(V^\vee)) \arrow[r, "p_*"] & S(W^\vee, \mathrm{Haar}(W^\vee)) \end{tikzcd} \qquad \begin{tikzcd} S^*(V) & S^*(W) \arrow[l, "i_*"'] \\ \mathcal{G}(V^\vee) \arrow[u, "\mathcal{F}"] & \mathcal{G}(W^\vee). \arrow[l, "p^*"'] \arrow[u, "\mathcal{F}"'] \end{tikzcd}$$

Note that this is possible since p is a submersion (linear and surjective) so pushing Schwartz measures along it yields Schwartz measures.

Proof. We start by showing the right hand side diagram commutes. Since i_*, the Fourier transform and p^* are continuous with respect to the weak topology, it is enough to prove commutativity for a dense set in $S^*(W)$.
First take the delta function $\delta_0 \in S^*(W)$, it is a compactly supported measure, and it holds that $i_*(\delta_0) = \delta_0$. Furthermore, since $\mathcal{F} : S^*(V) \to \mathcal{G}(V^\vee)$ is defined via

duality we have that $\mathcal{F}(\delta_0) = 1$:

$$\langle \mathcal{F}(\delta_0), f\mu \rangle = \langle \delta_0, \mathcal{F}(f\mu) \rangle = F(f\mu)(0_{V^{\vee\vee}}) = \int\limits_{V^\vee} f d\mu = \langle 1, f\mu \rangle,$$

where the third equality is sensible since $\mathcal{F}(f\mu) \in S(V^{\vee\vee})$ and $0_{V^{\vee\vee}}(\chi) = 1$ for all $\chi \in V^\vee$. We can also show that $p^*(1) = 1$. Consider $\mathcal{G}(W^\vee)$ as a subspace of $C^{-\infty}(W^\vee)$, there the generalized Schwartz function 1 is a smooth function, and note that the following diagram, where the horizontal arrows are the inclusions is commutative:

$$\begin{tikzcd}
\mathcal{G}(V^\vee) \arrow[r, hook] \arrow[d, "p^*"'] & C^{-\infty}(V^\vee) \arrow[d, "p^*"'] & C^\infty(V^\vee) \arrow[l, hook'] \arrow[d, "p^*"'] \\
\mathcal{G}(W^\vee) \arrow[r, hook] & C^{-\infty}(W^\vee) & C^\infty(W^\vee). \arrow[l, hook']
\end{tikzcd}$$

Now, note that every measure $f\mu \in \mu_c^\infty(V^\vee)$ can be treated either as a functional on smooth functions (since it has compact support as a distribution), or as the parameter a generalized function takes values on. This is utilized in the second equality bellow to yield the required result:

$$\langle p^*_{C^{-\infty}}(1), f\mu \rangle = \langle 1, p_*(f\mu) \rangle = \langle p_*(f\mu), 1 \rangle = \langle f\mu, p^*_{C^\infty}(1) \rangle = \langle f\mu, 1 \rangle = \langle 1, f\mu \rangle.$$

Note that since p_* is a submersion pushing forward a compactly supported smooth measure along it yields a smooth measure.

Since δ_w for any $w \in W$ is just a translation of δ_0 by w, its Fourier transform is $\mathcal{F}(\delta_w)(\chi) = \chi(w)$, and i_* and p^* are invariant to translations, the diagram is commutative for delta distributions. The space of delta distributions $\mathrm{span}_{\mathbb{C}}\{\delta_w\}_{w \in W}$ is dense w.r.t the weak topology since for every function f with $f(x_0) \neq 0$ we can take suitable $c \in \mathbb{C}$ such that $|\langle \xi - c\delta_x, f \rangle|$ is small as desired.

To see this implies the commutativity of the left diagram, it is enough to show that if $A^* = 0$ for $A^* : V_2^* \to V_1^*$ where A^* is the dual map to the linear map $A : V_1 \to V_2$, then $A = 0$, and use this for $\mathcal{F}i_* - p^*\mathcal{F}$.

If $A^* = 0$, we have for every $\xi_2 \in V_2^*$ and $v_1 \in V_1$ that $0 = \langle A^*\xi_2, v_1 \rangle = \langle \xi_2, Av_1 \rangle$. If there exists $v_1 \in V_1$ such that $Av_1 \neq 0$, then we can define a non-zero linear functional $\xi : \mathrm{span}_{\mathbb{C}}\{Av_1\} \to \mathbb{C}$ via $\langle \xi, Av_1 \rangle = 1$, and extend it to a non-zero continuous functional $\xi_2 \in V_2^*$ by the Hahn-Banach theorem. This yields a contradiction as

$$1 = \langle \xi_2, Av_1 \rangle = \langle A^*\xi_2, v_1 \rangle = \langle 0, v_1 \rangle = 0.$$

$\square$

10. Wave-front set

The wave-front set of a generalized function ξ is the collection of all points and codirections in which ξ is not smooth. It is a very important invariant of the generalized function. For example, there are some operations on functions, like product or pull-back, that do not extend to arbitrary generalized functions but do extend to generalized functions under some conditions on the wave-front set. The term comes from physics. Every differential equation satisfied by a generalized function will give a restriction on its wave-front set.

Example 10.0.1. *Let ξ be the δ-function of the x-axis on $\mathbb{R}^2$, i.e. the generalized function given by $\langle \xi, f dxdy\rangle := \int f(x)dx\rangle$, let $L \subset \mathbb{R}^2$ denote the x-axis and $L^\perp \subset (R^2)^*$ denote the subspace of functionals vanishing on L. Then $\mathrm{WF}(\xi) = L \times L^\perp$.*

We will now define the wave-front set by characterizing properties.

Definition 10.0.2. *The wave-front set is an assignment of a closed subset $\mathrm{WF}(\xi) \subset T^*M$ for any $\xi \in C^{-\infty}(M, E)$ such that*

(1) *For any isomorphism $\nu : (M, E) \simeq (M', E')$,*
$$\tilde{\nu}(\mathrm{WF}(\xi)) = \mathrm{WF}(\nu_*(\xi)).$$
(2) $\mathrm{WF}(\xi)$ *is conical in the cotangent directions, i.e.*
$$(x, v \in \mathrm{WF}(\xi) \Rightarrow (x, \lambda v) \in \mathrm{WF}(\xi) \quad \forall \lambda \in F.$$
(3) $p_M(\mathrm{WF}(\xi)) = \mathrm{WF}(\xi) \cap (M \times \{0\}) = \mathrm{supp}(\xi)$,
where $p_M : T^*M \to M$ denotes the natural projection.
(4) $\xi \in C^\infty(M, E) \Leftrightarrow \mathrm{WF}(\xi) \subset M \times \{0\}$
(5) $\mathrm{WF}(f\xi + g\eta) \subset \mathrm{WF}(\xi) \cup \mathrm{WF}(\eta)$
(6) *For another bundle E' over M, let $\eta \in C^{-\infty}(M, E)$ and consider $\xi \oplus \eta \in C^{-\infty}(M, E \oplus E')$. Then $\mathrm{WF}(\xi \oplus \eta) = \mathrm{WF}(\xi) \cup \mathrm{WF}(\eta)$.*
(7) *For any open subset $U \subset M$, $\mathrm{WF}(\xi|_U) = \mathrm{WF}(\xi) \cap p_M^{-1}(U)$.*
(8) *Let $\nu : N \to M$ be a submersion. Then $\mathrm{WF}(\nu^*\xi) = \nu^*(\mathrm{WF}(\xi))$, where the operation ν^* on subsets of cotangent bundles will be defined below.*
(9) *Let $\nu : M \to N$ be a smooth map such that $\nu|_{\mathrm{supp}(\xi)}$ is proper. Then $\mathrm{WF}(\nu_*(\xi)) \subset \nu_*(\mathrm{WF}(\xi))$, where the operation ν_* on subsets of cotangent bundles will be defined below.*

Definition 10.0.3. *(i) For sets A, B, and subsets $X \subset A, Y \subset B$, $S \subset A \times B$ define subsets $S_*(X) \subset B$ and $S^*(Y) \subset A$ by*
$$S_*(X) = \{y \in B \,|\, \exists x \in X \, s.t. (x, y) \in S\} \quad S^*(Y) = \{x \in A \,|\, \exists y \in Y \, s.t. (x, y) \in S\}.$$
(ii) For a morphism smooth map $\nu : M \to N$, define $\Delta_\nu \subset T^*M \times T^*N$ by
$$\Delta_\nu := \{((m, v), (n, w) \in T^*M \times T^*N \,|\, \nu(m) = n, \, d_m^*\nu(w) = v\}.$$

For a subset $X \subset T^*M$ define $\nu_*(X) := (\Delta_\nu)_*(X) \subset T^*N$ and for a subset $Y \subset T^*N$ define $\nu^*(Y) := (\Delta_\nu)^*(Y) \subset T^*M$.

Exercise 10.0.4 (*)**.** *Check whether the wave-front set is uniquely defined by the above properties.*

Hint. By (7), WF is local and thus it is enough to prove for $\xi \in C^{-\infty}(F^n)$. Let $(0,v) \notin \mathrm{WF}(\xi)$, and let $\rho \in C_c^\infty(F^n)$ be constant 1 in a neighborhood of zero. Consider $v_*(\rho\xi) \in C^{-\infty}(F)$. From (9) and (4), $v_*(\rho\xi) \in C^\infty(F)$. Thus, if $v_*(\rho\xi) \notin C^\infty(F)$ for some ρ as above then $(0,v) \in \mathrm{WF}(\xi)$.
Let us now find a necessary condition for $(0,v) \in \mathrm{WF}(\xi)$. We will use the Radon transform, which maps $f \in C_c^\infty(V)$ to the integrals of f on all affine hyperplanes, *i.e.* hyperplanes not necessarily passing through the origin. For a vector space W, denote $\bar{W} := \mathbb{P}(W \oplus F)$. Note that this is a compact manifold and that the manifold of all affine hyperplanes in V is $\overline{V^*}$, and the manifold of all affine lines in V is $\bar{V}$. Define $R := \{(l,H) \in \bar{V} \times \overline{V^*} \,|\, l \in H\}$, and let $p_1 : R \to \bar{V}$ and $p_2 : R \to \overline{V^*}$ be the projections. Then the Radon transform is $(p_2)_* \circ p_1^*$. It is known that this transform is invertible, and thus any distribution is the Radon transform of another one. Thus, the conditions (9) and (8) give an upper bound on WF. Hopefully, the two bounds together determine WF. □

10.1. **Definition of the wave-front set.** Let us now give a constructive definition of WF, following Hörmander, and then prove the above properties. It will take several steps.

Definition 10.1.1. *For any local field F, let V be an F-vector space, $v \in V$ and $f \in C^\infty(V)$. We say that f* vanishes asymptotically *along v if there exists an open neighborhood U of v and $\rho \in C_c^\infty(U)$ such that $p^*(\rho)m^*f \in \mathcal{S}(U \times F)$, where $m : V \times F \to V$ is given by $m(v,\lambda) := \lambda v$ and $p : V \times F \to F$ is the projection.*

Exercise 10.1.2. *TFAE:*

(i) *f vanishes asymptotically along v for any $v \neq 0 \in V$*
(ii) *$f \in \mathcal{S}(V)$.*

Exercise 10.1.3. *Show that f vanishes asymptotically along 0 if and only if $f = 0$.*

Exercise 10.1.4. *For any Lie group G, we have $\mathcal{F}(Dist_c(G)) \subset C^\infty(\check{G})$.*

Definition 10.1.5. *Let V be a vector space, and $\xi \in Dist(V)$.*

(i) *We say that ξ is smooth at $(x,w) \in V \times V^*$ if there exists $\rho \in C_c^\infty(V)$ such that $\rho(x) = 1$ and $\mathcal{F}(\rho\xi)$ vanishes asymptotically along w.*
(ii) $\mathrm{WF}(\xi) = \{(x,w) \in V \times V^* \,|\, \xi$ *is not smooth at* $(x,w)\}$.
(iii) *For $x \in V$, let $\mathrm{WF}_v(\xi) := \mathrm{WF}(\xi) \cap \{x\} \times V^*$*

Theorem 10.1.6 (The proof is complicated)**.** *Let $\nu : V \to V$ be a diffeomorphism s.t. $\nu(0) = 0$ and $d_0\nu = Id$. Then $\mathrm{WF}_0(\nu^*(\xi)) = \mathrm{WF}_0(\xi)$ for any $\xi \in Dist(V)$.*

Exercise 10.1.7 (*). *A distribution ξ on V is smooth at $(x,w) \in V \times V^*$ if and only if for any $\rho \in C_c^\infty(V)$ with $\rho(x)=1$, the Fourier transform $\mathcal{F}(\rho\xi)$ vanishes asymptotically along w.*

Corollary 10.1.8. *Let $\nu : V \to V$ be a diffeomorphism. Then*

$$\mathrm{WF}(\nu^*(\xi)) = \nu^*(\mathrm{WF}((\xi))).$$

Corollary 10.1.9. *The definition of* WF *extends to generalized sections of vector bundles on manifolds.*

Exercise 10.1.10. *Let $L \subset V$ be linear spaces of dimensions 1 and 2. Fix a Haar measure μ on L and define $\delta_L \in \mathrm{Dist}(V)$ by $\langle \delta_L, f\rangle := \int_L f\mu$. Compute* $\mathrm{WF}(\delta_L)$.

Exercise 10.1.11. *Properties* (1)-(7) *of* WF *hold.*

Exercise 10.1.12. *For any epimorphism of Lie groups $p : G \twoheadrightarrow H$, and any $\xi \in Dist_c(G)$, we have $\mathcal{F}(p_*(\xi)) = \mathcal{F}(\xi)|_{\check{H}}$.*

Exercise 10.1.13. *Property* (8) *holds.*

Exercise 10.1.14. *Property* (9) *holds.*

Hint. Any map $\nu : M \to N$ decomposes as a composition of the closed embedding $graph(\nu) : M \hookrightarrow M \times N$ and the projection $M \times N \twoheadrightarrow N$. Thus it is enough to prove for a closed embedding and for a projection. By the locality of WF and invariance to isomorphisms it is enough to prove for a linear embedding and a linear epimorphism. ☐

10.2. **Advanced properties of the wave-front set.**

Definition 10.2.1. *Let $\Gamma \subset T^*M$ be a closed subset. Define*

$$C_\Gamma^{-\infty}(M) := \{\xi \in C^{-\infty}(M) \,|\, \mathrm{WF}(\xi) \subset \Gamma\}.$$

Definition 10.2.2. *(i) For a non-archimedean F, an F-analytic manifold M and an F-vector space W define*

$$\mathcal{S}^W(M) := \{f \in C^\infty(M \times W) \,|\, pr_M|_{\mathrm{supp}(f)} \text{ is proper }\}.$$

(ii) For a smooth manifold M and an $\mathbb{R}$-vector space W define

$$\mathcal{S}^W(M) := \{f \in C^\infty(M \times W) \,|\, \forall \text{ compact } K \subset M, \forall m,n \in \mathbb{N}^{\dim W}$$
$$\forall D \in Diff(M). \sup_{(x,w)\in K\times W} ||Df^{(n)}_{x,w} w^m|| < \infty\}.$$

Definition 10.2.3. *Define topology on $C_\Gamma^{-\infty}(V)$ by: $\xi_i \to \xi$ if $\xi \xrightarrow{w} \xi$ in $C^{-\infty}(V)$ and for any $v \in V$ there exists $\epsilon > 0$ and $\rho \in C_c^\infty(B_\epsilon(v))$ such that for any $l \in V^*$ we have*

$$m^*\mathcal{F}(\rho\xi_i)|_{B_\epsilon(l)\times F} \to m^*\mathcal{F}(\rho\xi_i)|_{B_\epsilon(l)\times F}$$

in $\mathcal{S}^F(B_\epsilon(l) \times F)$.

Definition 10.2.4. *For a smooth map $\nu : M \to N$, define*

$$S_\nu := \{(\nu(x), w) \in T^*N \mid d_x\nu^*(w) = 0\}.$$

Note that $S_\nu = \nu_*(M \times \{0\})$.

Theorem 10.2.5. *Let $\nu : M \to N$ and let $\Gamma \subset T^*N$ be a closed subset such that*

$$\Gamma \cap S_\nu \subset N \times \{0\}.$$

Then $\nu^* : C^\infty(N) \to C^\infty(M)$ can be continuously extended to

$$\nu^* : C_\Gamma^{-\infty}(N) \to C_{\nu^*(\Gamma)}^{-\infty}(M).$$

Proof. For a submersion, one can pullback any distribution, $C_\Gamma^{-\infty}(N) = C^\infty(N)$. Also, the statement is local. Thus, enough to prove it for $\nu : V \to W$. Any such ν decomposes to $i : V \to V \times W$ given by $i(v) := (v, 0)$, $\nu' : V \times W \to V \times W$ given by $(v, w) \mapsto (v, \nu(v) + w)$ and the projection on W. Thus, it is enough to prove for i. For i we have

$$S_i = CN(V) = V \times W^* \subset (V \oplus W) \times (V^* \oplus W^*) = T^*(V \oplus W).$$

Also, enough to prove for the case $\dim W = 1$. This we do by twisting pushforward by $\mathcal{F}$.
More precisely, we use Theorem 9.0.26 on functoriality of Fourier transform. By this theorem, for $f \in \mathcal{S}(V \otimes W)$ we have $\mathcal{F}(i^*f) = p_*(\mathcal{F}(f))$, where $p : V^* \oplus W^* \to V^*$ is the projection. This formula extends to $\xi \in C_{V \times W^*}^{-\infty}(V \oplus W)$, since $\mathcal{F}(\rho\xi)$ vanishes asymptotically along W^* for any $\rho \in C_c^\infty(V \oplus W)$. More precisely, we can choose a sequence $\rho_n \in C_c^\infty(V \oplus W)$ that on every compact becomes the constant 1 starting from some index, and define $\nu^*\xi$ to be the limit of $\mathcal{F}^{-1}(p_*(\mathcal{F}(\rho_n\xi)))$. $\square$

Corollary 10.2.6. *Let $\xi, \eta \in C^{-\infty}(M)$ such that* $\mathrm{WF}(\xi) \cap \mathrm{WF}(\eta) \subset M \times \{0\}$. *Then we can define the product $\xi \cdot \eta \in C_{\mathrm{WF}(\xi)+\mathrm{WF}(\eta)}^{-\infty}(N)$.*

Proof. Define $\Delta : M \to M \times M$ by $\Delta(m) := (m, m)$. Then $\xi \cdot \eta = \Delta^*(\xi \otimes \eta)$. $\square$

Theorem 10.2.7. *Let $\xi \in C^{-\infty}(V)$ and let $Z \subset V^*$ be a closed conical set such that* $\mathrm{supp}\mathcal{F}(\xi) \subset Z$. *Then* $\mathrm{WF}(\xi) \subset V \times Z$.

Intuitively, this theorem makes sense since the cotangent directions in the wave-front set form the asymptotic support of the Fourier transform. Let us now give the proof in the p-adic case, since the proof in the real case is similar, though longer.

Proof. Enough to show that for any $\rho \in C_c^\infty(V)$, $\mathcal{F}(\rho\xi)$ is asymptotically supported in Z. Note that $\mathcal{F}(\rho\xi) = \mathcal{F}(\rho) * \mathcal{F}(\xi)$ and thus $\mathrm{supp}\mathcal{F}(\rho\xi) \subset \mathrm{supp}\mathcal{F}(\rho) + Z$. Since $\mathrm{supp}\mathcal{F}(\rho)$ is a compact set, $\mathcal{F}(\rho\xi)$ is eventually zero in any direction not in Z. Since Z is closed, this implies that $\mathcal{F}(\rho\xi)$ is asymptotically supported in Z and thus $\mathrm{WF}(\xi) \subset V \times Z$. $\square$

Theorem 10.2.8. *Let M be a smooth manifold, D be a differential operator on M, and $\xi \in C^{-\infty}(M)$ such that $D\xi = 0$. Then* $\mathrm{Symb}(D)(\mathrm{WF}(xi)) = \{0\}$.

Sketch of proof. Since the question is local, we can assume that M is a vector space V. Fix $x \in V$ and let p be the polynomial on V^* given by $p(l) := \mathrm{Symb}(D)(v, l)$. Let $l \in V^*$ be a cotangent direction such that $p(l) \neq 0$. We want to show that $p \notin \mathrm{WF}_x(\xi)$. Define

$$\nu(\xi, l) := \sup_{w \in l + B_\epsilon(0)} \overline{\lim_{\lambda \to \infty}} \frac{\ln |\mathcal{F}(\xi)(\lambda w)|}{\ln |\lambda|}$$

This is the order of asymptotics as $\lambda \to \infty$ of $\mathcal{F}(\xi)$ near the direction l. It is not $+\infty$ since $\mathcal{F}(\xi)$ is a tempered distribution. We want to show that $\nu(\xi, l) = -\infty$ for ϵ small enough. Denote $\psi(\alpha) := \exp(2\pi i \alpha)$, and define $f_\lambda \in C^\infty(V)$ by $f_\lambda(v) := \psi(\lambda l(v))$. We have

$$0 = \langle \rho D\xi, f_\lambda \rangle = \langle \xi, D(\rho f_\lambda) \rangle.$$

Using the Leibnitz rule we get

$$D(\rho f_\lambda)(v) = D(\rho(v)\psi(\lambda l(v))) = \sum_{m=0}^{degD} \lambda^m \psi(\lambda l(v))\rho_m = \sum_{m=0}^{degD} \lambda^m f \rho_m$$

for some collection $\rho_m \in C_c^\infty(V)$. Thus

$$0 = \langle \xi, D(\rho f_\lambda) \rangle = \sum \lambda^m \langle \xi, f\rho_m \rangle = \sum \lambda^m \mathcal{F}(\rho_m \xi)(\lambda l) = \sum \lambda^m (\mathcal{F}(\xi) * \mathcal{F}(\rho_m))(\lambda l).$$

Thus

$$\mathcal{F}(\xi) * \mathcal{F}(\rho_{degD})(\lambda l) = -\lambda^{-1}(\mathcal{F}(\xi) * \mathcal{F}(\rho_{degD-1}))(\lambda l) + \ldots$$

and thus $\nu(\xi, l) = \nu(\xi, l) - 1$. Thus $\nu(\xi, l) = -\infty$. $\square$

10.3. **Sketch of proof of the invariance to isomorphisms in the p-adic case.** Let $V := F^n$. Let $\psi : F \to U(1) \subset C^\times$ be a non-trivial unitary additive character with $\psi(B_1(0)) = \{1\}$. Identify $\check{V}$ with V^* using ψ.

Notation 10.3.1. *Let $l \in V^*$ and $r, \varepsilon > 0 \in \mathbb{R}$. Denote*

$C^{-\infty}_{l,r,\varepsilon}(V) := \{\xi \in C^{-\infty}(V) \,|\, \mathrm{supp}(\xi) \subset B_r(0)$ *and* $m^*(\mathcal{F}(\xi))|_{B_\varepsilon(l) \times F} \in \mathcal{S}^F(B_\varepsilon(l) \times F)\}$

$C^{-\infty}_{l,r,\varepsilon,\alpha}(V) := \{\xi \in C^{-\infty}(V) \,|\, \mathrm{supp}(\xi) \subset B_r(0)$ *and* $\mathrm{supp}(m^*(\mathcal{F}(\xi)))|_{B_\varepsilon(l) \times F} \subset B_\varepsilon \times B_\alpha(F)\}$

Note that $C^{-\infty}_{l,\varepsilon}(V) := \cup_{\alpha \in \mathbb{R}} C^{-\infty}_{l,\varepsilon,\alpha}(V)$.

Exercise 10.3.2. *(i) If $\xi \in C^{-\infty}_{l,r,\varepsilon,\alpha}(V)$ with $||l|| = 1$ then $1_{B_\delta(l)}\xi \in C^{-\infty}_{l,r,\varepsilon,\alpha+\delta^{-1}}(V)$.*
(ii) If $\xi \in C^{-\infty}_{l,r,\varepsilon}(V)$ and $\rho \in C_c^\infty(V)$ then $\rho\xi \in C^{-\infty}_{l,r,\varepsilon}(V)$.

Notation 10.3.3. *For a diffeomorphism $\nu : V \to V$ and $x \in V$ denote by $Aff_x\nu$ the affine approximation to ν at x. Namely*

$$Aff_x\nu(y) := \nu(x) + d_x\nu(y - x).$$

Exercise 10.3.4. *Let $\nu : V \to V$ be a diffeomorphism s.t. $\nu(0) = 0$ and $d_0\nu = Id$. Let $r > 0$. Show that $\exists C \, \forall \delta > 0 \, \forall x \in B_r(0)$ we have*

$$\sup_{B_\delta(x)} ||\nu - Aff_x\nu|| < C\delta^2.$$

Hint. Use Taylor series □

Exercise 10.3.5. *Let $\xi \in \mathcal{S}_c^*(V)$. Let $\nu_1, \nu_2 : V \to V$ and let $\varepsilon > 0$ be s.t.*

$$\sup_{\mathrm{supp}\xi} ||\nu_1 - \nu_2|| < \varepsilon.$$

Then

$$\mathcal{F}(\nu_1^*\xi)|_{B_{\varepsilon^{-1}}(0)} = \mathcal{F}(\nu_2^*\xi)|_{B_{\varepsilon^{-1}}(0)}.$$

Exercise 10.3.6. *Let $r, \varepsilon, \alpha > 0$ with $\varepsilon < 1$ and let $l \in V^*$ s.t. $||l|| = 1$. Let $A : V \to V$ be an affine transformation s.t. $||d_0A - Id|| < 1$ and $A(B_r(0)) \subset B_r(0)$. Then*

$$A^*(C^{-\infty}_{l,r,\varepsilon,\alpha}(V)) \subset C^{-\infty}_{d_0^*A(l),r,\varepsilon,\alpha}(V).$$

This exercise follows from the rules for conjugation of linear transformations by Fourier transform.
The theorem follows now from the following specialization.

Theorem 10.3.7. *Let $r, \varepsilon, \alpha > 0$ with $\varepsilon < 1$ and let $l \in V^*$ s.t. $||l|| = 1$. Let $\nu : V \to V$ be a diffeomorphism such that*

1. *$\nu(0) = 0$*
2. *$||d_x\nu - Id|| < \varepsilon$*
3. *for any $x \in B_r(0)$ and any $r' \leq r$ we have $\nu(B_{r'}(x)) = B_{r'}(\nu(x))$.*

Let $\xi \in C^{-\infty}_{l,r,\varepsilon,\alpha}(V)$. Then

$$\nu^*\xi \in C^{-\infty}_{l,r,\varepsilon}(V).$$

Proof. Let C be as in Exercise 10.3.4. Fix $\lambda \in F$ with $|\lambda| > \alpha + C + \sqrt{C(C + 2\alpha)}$. Let $\delta = 1/(\sqrt{2\lambda C})$. Thus $\alpha + \delta^{-1} < |\lambda| < C^{-1}\delta^{-2}$. It is enough to show that for any $l' \in B_\varepsilon(l)$ we have $\mathcal{F}(\nu^*\xi)(\lambda l') = 0$. Present $B_r(0)$ as a disjoint union of balls $U_i = B_\delta(x_i)$ of radius δ. We have

$$\mathcal{F}(\nu^*\xi)(\lambda l') = \sum_i \mathcal{F}(\nu^*(1_{U_i}\xi))(\lambda l').$$

Since $|\lambda| < C^{-1}\delta^{-2}$, Exercises 10.3.4 and 10.3.5 imply

$$\mathcal{F}(\nu^*(1_{U_i}\xi))(\lambda l') = \mathcal{F}((Aff_{x_i}\nu)^*(1_{U_i}\xi))(\lambda l').$$

By Exercise 10.3.2 (i) and by the assumptions $1_{U_i}\xi \in C^{-\infty}_{l,r,\varepsilon,\alpha+\delta^{-1}}(V)$. By Exercise 10.3.6 this implies

$$(Aff_{x_i}\nu)^*(1_{U_i}\xi) \in C^{-\infty}_{d_{x_i}^*\nu(l),r,\varepsilon,\alpha+\delta^{-1}}(V).$$

Since $|\lambda| > \alpha + \delta^{-1}$ and $l' \in B_\varepsilon(d^*_{x_i}\nu(l))$ we have

$$\mathcal{F}((Aff_{x_i}\nu)^*(1_{U_i}\xi))(\lambda l') = 0 \quad \forall i.$$

Summarizing, we have

$$\mathcal{F}(\nu^*\xi)(\lambda l') = \sum_i \mathcal{F}(\nu^*(1_{U_i}\xi))(\lambda l') = \sum_i \mathcal{F}((Aff_{x_i}\nu)^*(1_{U_i}\xi))(\lambda l') = 0.$$

□

Remark 10.3.8. *(1) The method in the proof is called the stationary phase method, which is a central method in microlocal analysis.*

(2) In order to use the affine approximation we decomposed $\xi = \sum_i 1_{U_i}\xi$. The multiplication of ξ by 1_{U_i} "damaged" ξ but this "damage" can be controlled using 10.3.2(i), and is apparently moderate since the affine approximation is good to the second order.

11. The Weil representation, the oscillator representation and an application

In this section we show that the Fourier transform is not alone - it is part of an infinite group of operators. This group is a representation of a double cover of $\mathrm{SL}_2(F)$. Thus this section requires some knowledge of representation theory. To motivate the existence of this representation we first describe the Heisenberg group and its representations.

Definition 11.0.1. *Let $V := F^n$ and let ω be the standard symplectic form on $W_n := V \oplus V^*$. The Heisenberg group H_n is the algebraic group with underlying algebraic variety $W_n \times F$ with the group law given by*

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + 1/2\omega(w_1, w_2)).$$

Define a unitary character χ of $\mathbb{R}$ by $\chi(z) := \exp(2\pi iz)$.

Definition 11.0.2. *The oscillator representation of H_n is given on the space $L^2(V)$ by*

$$(\sigma(x, \varphi, z)f)(y) := \chi(\varphi(y) + z))f(x + y). \tag{3}$$

Note that the center of H_n is $0 \times F$, and it acts on σ by the character χ, which can be trivially extended to a character of $V^* \times F$.
It is easy to see that σ is the unitary induction of (the extension of) the character χ from $V^* \times \mathbb{R}$ to $H_n = (V \oplus V^*) \times F$.

Lemma 11.0.3. *The space of smooth vectors in σ is $\mathcal{S}(V)$, and the Lie algebra of H_n acts on it by*

$$\sigma(v)f := \partial_v f,\ \sigma(\varphi)f := \varphi f,\ \sigma(z)f := 2\pi izf. \tag{4}$$

Proof. Formula (4) is obtained from (3) by derivation. Now, it is known that the space of smooth vectors in a unitary induction consists of the smooth L^2 functions whose derivatives also lie in L^2. □

Theorem 11.0.4 (Stone-von-Neumann)**.** *The oscillator representation σ is the only irreducible unitary representation of H_n with central character χ.*

Idea of the proof. Let me ignore all the analytic difficulties. Consider the normal commutative subgroup $A := V \times F$. Conjugation in H_n defines an action of V on the dual group of A. This action has only two orbits. The closed orbit is the singalton $\{1\}$ and the open orbit $\mathcal{O}$ is the complement to the closed one. The restriction $\sigma|_A$ decomposes to a direct integral of characters in $\mathcal{O}$, each "with multiplicity one". The restriction of any non-zero subrepresentation $\rho \subset \sigma$ to A will also include χ, and thus the whole orbit $\mathcal{O}$ of χ. Thus $\rho = \sigma$ and σ is irreducible.
Now let τ be any irreducible unitary representation of H_n with central character χ. Then the restriction of τ to A will again include all the characters in $\mathcal{O}$ with multiplicity one. Thus τ is the induction of an irreducible representation of the stabilizer of χ in H_n. However, this stabilizer is A and thus $\tau \simeq \sigma$. □

Note that the symplectic group $\mathrm{Sp}(V \oplus V^*)$ acts on H_n by automorphisms, preserving the center. Thus the theorem implies the following corollary.

Corollary 11.0.5. *For every $g \in \mathrm{Sp}(V \oplus V^*)$ there exists a (unique up to a scalar multiple) linear automorphism T of $\mathcal{S}(V)$ such that For any $h \in H_n$ we have $\sigma(h^g) = T\sigma(h)T^{-1}$.*

The uniqueness part of Corollary 11.0.5 follows from Schur's lemmas. Corollary 11.0.5 defines a projective representation of $\mathrm{Sp}(V \oplus V^*)$ on $\mathcal{S}(V)$, i.e. a map $\tau : \mathrm{Sp}(V \oplus V^*) \to \mathrm{GL}(\mathcal{S}(V))$ such that $\tau(gh) = \lambda_{g,h}\tau(g)\tau(h)$. It is not possible to coordinate the scalars in order to obtain an honest representation of $\mathrm{Sp}(V \oplus V^*)$, but it is possible to obtain a representation of a double cover $\widetilde{\mathrm{Sp}}(V \oplus V^*)$, called the metaplectic group. This was shown by A. Weil. We will not give the formulas for the full Weil representation, but rather for its restriction to the subgroup $\widetilde{\mathrm{SL}_2}(F)$ embedded by

$$E_{11} \mapsto Id_V,\, E_{12} \mapsto T,\, E_{21} \mapsto T^{-1},\, E_{22} \mapsto Id_{V^*},$$

where by Id_V we mean the operator that is Id on V and zero on V^* and by T the operator that is zero on V^* and on V is given by ω.
Suppose that F is non-archimedean, and identify V with V^* using a non-degenerate quadratic form q. Also, identify σ with $\mathcal{S}(V)$. Let $n := \dim V$. Then the Weil representation is given by

$$\pi(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \varepsilon)f(v) = \varepsilon^n \psi(uq(v))f(v) \tag{5}$$

$$\pi(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \varepsilon)f(v) = \varepsilon^n |t|^{n/2}\gamma(q)\gamma(tq)^{-1}f(tv) \tag{6}$$

$$\pi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \varepsilon)f(v) = \varepsilon^n \gamma(q)\mathcal{F}(f)(v) \tag{7}$$

Here, $\gamma(q)$ is a certain eights root of unity that depends on the quadratic form, and ψ is the non-trivial unitary additive character that we use to identify $\check{V}$ with V^*. Note that this representation defines a representation of $\mathrm{SL}_2(F)$ if and only if n is even. The existence of the Weil representation and the formulas above imply the following corollary.

Corollary 11.0.6 (Rallis - Schiffmann). *Let $\xi \in \mathcal{S}(V)$ and let q be a non-degenerate quadratic form on V. Let Z denote the zeros of q in V. Let $\xi \in \mathcal{S}^*(V)$ such that* $\mathrm{supp}\xi \subset V$ *and* $\mathrm{supp}\mathcal{F}(\xi) \subset V$. *Then $\xi = \gamma(q)\mathcal{F}(\xi)$ and*

$$\langle \xi, f(tv)\rangle = |t|^{-n/2}\gamma(q)^{-1}\gamma(tq)\langle \xi, f(v)\rangle.$$

Moreover, if $\xi \neq 0$ then n is even.

Proof. Since $\mathrm{supp}\xi \subset Z$, we have $\psi(uq(v))\xi = \xi$ and thus $\pi(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \varepsilon)\xi = \varepsilon^n\xi$. Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathrm{supp}\mathcal{F}(\xi) \subset Z$, (7) and (5) imply $\pi(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \varepsilon)\xi = \varepsilon^n\xi$. Since the subgroups $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ generate $\mathrm{SL}_2(F)$, we get that $(g,\varepsilon)\xi = \varepsilon^n\xi$ for any $(g,\varepsilon) \in \widetilde{\mathrm{SL}_2}(F)$. We can assume $\xi \neq 0$. Then we get that n is even, and $\widetilde{\mathrm{SL}_2}(F)$ acts trivially on ξ. The lemma follows now from (6) and (7). $\square$

In the case $F = \mathbb{R}$, one can prove a slightly weaker lemma. The problem in generalizing the argument above to the archimedean case is that in this case the condition $\mathrm{supp}\mathcal{F}(\xi) \subset V$ does not imply $\psi(uq(v))\xi = 0$, but rather that there exists k such that $\psi(uq(v))^k\xi = 0$. On the other hand, in this case one can use the Lie algebra $\mathfrak{sl}_2$.

For the Archimedean version of the corollary we will need the following definition.

Definition 11.0.7. *For any $t \in F^\times$ denote by $\rho(t)$ the homotheties action on $Sc^*(V)$. Denote also $\delta(t) := \gamma(q)/\gamma(tq)$.*

*We say that a distribution $\xi \in \mathcal{S}^*(V)$ is **adapted to** q if for any $t \in F^\times$ we have*

$$\text{either } (i)\ \rho(t)\xi = \delta(t)|t|^{\dim V/2}\xi \quad \text{or} \quad (ii)\ \rho(t)\xi = \delta(t)t|t|^{\dim V/2}\xi.$$

Theorem 11.0.8. *Let $L \subset \mathcal{S}^*_V(Z(q))$ be a non-zero subspace such that for all $\xi \in L$ we have $\mathcal{F}_q(\xi) \in L$ and $q \cdot \xi \in L$ (here B is viewed as a quadratic function). Then there exists a non-zero distribution $\xi \in L$ which is adapted to q.*

Using it one obtains the following result.

Proposition 11.0.9. *Let V be a real vector space and let q be a non-degenerate quadratic form on V. Let Z denote the zeros of q in V. Let $\xi \in \mathcal{S}^*(V)$ such that* $\mathrm{supp}\xi \subset V$ *and* $\mathrm{supp}\mathcal{F}(\xi) \subset V$. *Then there exists a unitary character χ of $\mathbb{R}^\times$ and $m \in \{0,1\}$ either such that for any $t \in \mathbb{R}^\times$ we have*

$$\langle \xi, f(tv)\rangle = |t|^{-n/2-m}\chi(t)\langle \xi, f(v)\rangle.$$

12. Schwartz functions on Nash manifolds

12.1. **Semi-algebraic sets and the Seidenberg-Tarski theorem.** In this section we follow [BCR].

Definition 12.1.1. *A subset $A \subset \mathbb{R}^n$ is called a **semi-algebraic set** if it can be presented as a finite union of sets defined by a finite number of polynomial equalities and inequalities. In other words, if there exist finitely many polynomials $f_{ij}, g_{ik} \in R[x_1, ..., x_n]$ such that*

$$A = \bigcup_{i=1}^{r} \{x \in \mathbb{R}^n | f_{i1}(x) > 0, ..., f_{is_i}(x) > 0, g_{i1}(x) = 0, ..., g_{it_i}(x) = 0\}.$$

Lemma 12.1.2. *The collection of semi-algebraic sets is closed with respect to finite unions, finite intersections and complements.*

Example 12.1.3. *The semi-algebraic subsets of $\mathbb{R}$ are unions of finite number of (finite or infinite) intervals.*

In fact, a semi-algebraic subset is the same as a union of connected components of an affine real algebraic variety.

Definition 12.1.4. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semi-algebraic sets. A mapping $\nu : A \to B$ is called **semi-algebraic** iff its graph is a semi-algebraic subset of $\mathbb{R}^{m+n}$.*

Proposition 12.1.5. *Let ν be a bijective semi-algebraic mapping. Then the inverse mapping ν^{-1} is also semi-algebraic.*

Proof. The graph of ν is obtained from the graph of ν^{-1} by switching the coordinates. $\square$

One of the main tools in the theory of semi-algebraic spaces is the Tarski-Seidenberg principle of quantifier elimination. Here we will formulate and use a special case of it. We start from the geometric formulation.

Theorem 12.1.6. *Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset and $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the standard projection. Then the image $p(A)$ is a semi-algebraic subset of $\mathbb{R}^{n-1}$.*

By induction and a standard graph argument we get the following corollary.

Corollary 12.1.7. *An image of a semi-algebraic subset of $\mathbb{R}^n$ under a semi-algebraic map is semi-algebraic.*

Sometimes it is more convenient to use the logical formulation of the Tarski-Seidenberg principle. Informally it says that any set that can be described in semi-algebraic language is semi-algebraic. We will now give the logical formulation and immediately after that define the logical notion used in it.

Theorem 12.1.8 (Tarski-Seidenberg principle, see e.g.[BCR, Proposition 2.2.4])**.** *Let Φ be a formula of the language $L(\mathbb{R})$ of ordered fields with parameters in $\mathbb{R}$. Then there exists a quantifier - free formula Ψ of $L(\mathbb{R})$ with the same free variables $x_1, \ldots, x_n$ as Φ such that $\forall x \in \mathbb{R}^n, \Phi(x) \Leftrightarrow \Psi(x)$.*

Definition 12.1.9. *A **formula of the language of ordered fields with parameters in** $\mathbb{R}$ is a formula written with a finite number of conjunctions, disjunctions, negations and universal and existential quantifiers ($\forall$ and $\exists$) on variables, starting from atomic formulas which are formulas of the kind $f(x_1, \ldots, x_n) = 0$ or $g(x_1, \ldots, x_n) > 0$, where f and g are polynomials with coefficients in $\mathbb{R}$. The free variables of a formula are those variables of the polynomials which are not quantified. We denote the language of such formulas by $L(\mathbb{R})$.*

Notation 12.1.10. *Let Φ be a formula of $L(\mathbb{R})$ with free variables $x_1, \ldots, x_n$. It defines the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that satisfy Φ. We denote this set by S_Φ. In short,*

$$S_\Phi := \{x \in \mathbb{R}^n | \Phi(x)\}.$$

Corollary 12.1.11. *Let Φ be a formula of $L(\mathbb{R})$. Then S_Φ is a semi-algebraic set.*

Proof. Let Ψ be a quantifier-free formula equivalent to Φ. The set S_Ψ is semi-algebraic since it is a finite union of sets defined by polynomial equalities and inequalities. Hence S_Φ is also semi-algebraic since $S_\Phi = S_\Psi$. $\square$

Proposition 12.1.12. *The logical formulation of the Seidenberg-Tarski principle implies the geometric one.*

Proof. Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset, and $pr : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the standard projection. Then there exists a formula $\Phi \in L(\mathbb{R})$ such that $A = S_\Phi$. Then

$pr(A) = S_\Psi$ where

$$\Psi(y) = \text{“}\exists x \in \mathbb{R}^n\, (pr(x) = y \wedge \Phi(x))\text{”}.$$

Since $\Psi \in L(\mathbb{R})$, the proposition follows now from the previous corollary. $\square$

In fact, it is not difficult to deduce the logical formulation from the geometric one. Let us now demonstrate how to use the logical formulation of the Seidenberg-Tarski theorem.

Corollary 12.1.13. *The closure of a semi-algebraic set is semi-algebraic.*

Proof. Let $A \subset \mathbb{R}^n$ be a semi-algebraic subset, and let $\overline{A}$ be its closure. Then $\overline{A} = S_\Psi$ where

$$\Psi(x) = \text{“}\forall \varepsilon > 0\, \exists y \in A\, |x-y|^2 < \varepsilon\text{”}.$$

Clearly, $\Psi \in L(\mathbb{R})$ and hence $\overline{A}$ is semi-algebraic. $\square$

Corollary 12.1.14. *The derivative f' of any differentiable semi-algebraic function $f : \mathbb{R} \to \mathbb{R}$ is semi-algebraic.*

Proof. The graph of f' equals S_Ψ, where

$$\Psi(x,y) = \text{“}\forall \varepsilon > 0\, \exists \delta > 0, \text{ s.t. } \forall 0 \neq \delta' \in (-\delta, \delta) \text{ we have } (f(x+\delta') - f(x) - y\delta')^2 < \varepsilon\delta')^2\text{”}.$$

Clearly, $\Psi \in L(\mathbb{R})$ and hence f' is semi-algebraic. $\square$

Corollary 12.1.15.
(i) The composition of semi-algebraic mappings is semi-algebraic.
(ii) The $\mathbb{R}$-valued semi-algebraic functions on a semi-algebraic set A form a ring, and any nowhere vanishing semi-algebraic function is invertible in this ring.
(iii) Images and preimages of semi-algebraic sets under semi-algebraic mappings are semi-algebraic.

Proposition 12.1.16. [BCR, Proposition 2.4.5] *Any semi-algebraic set in $\mathbb{R}^n$ has a finite number of connected components.*

Remark 12.1.17. *Over a non-archimedean local field F (e.g. $F = \mathbb{Q}_p$) one considers sets that are finite unions of finite intersections of sets of the form*

$$\{x \in F^n \text{ s.t. } p(x) \text{ is a } k\text{-th power}\}, \text{ or } \{x \in F^n \text{ s.t. } p(x) = 0.$$

An analog of the Seidenberg - Tarski theorem holds for such sets.

12.2. **Nash manifolds.** Let us now define the category of Nash manifolds, i.e. smooth semi-algebraic manifolds. I like this category since the Nash manifolds behave as tamely as algebraic varieties (e.g. posses some finiteness properties, and admit an analog of Hironaka's desingularization theorem), and in addition their

local structure is almost as easy as that of differentiable manifolds. In particular, they are locally trivial, and analogs of the implicit function theorem and the tubular neighborhood hold for them. The only thing we “loose" is the partition of unity.
Nash has shown that any compact smooth manifold has a unique structure of an affine Nash manifold. It was later shown that it also has uncountably many structures of non-affine Nash manifold. It is Artin-Mazur who first used the term of Nash manifold. They gave a fundamental theorem which states that an affine Nash manifold can be imbedded in a Euclidean space so that the image contains no singular points of its Zariski closure.
In this section we follow [BCR, Shi].

Definition 12.2.1. *A* ***Nash map*** *from an open semi-algebraic subset U of $\mathbb{R}^n$ to an open semi-algebraic subset $V \subset \mathbb{R}^m$ is a smooth (i.e. infinitely differentiable) semi-algebraic function. The ring of $\mathbb{R}$-valued Nash functions on U is denoted by $\mathcal{N}(U)$. A* ***Nash diffeomorphism*** *is a Nash bijection whose inverse map is also Nash.*

As we are going to do semi-algebraic differential geometry, we will need a semi-algebraic version of implicit function theorem.

Theorem 12.2.2 (Implicit Function Theorem for Nash manifolds, see e.g. [BCR, Corollary 2.9.8])**.** *Let $(x^0, y^0) \in \mathbb{R}^{n+p}$, and let $f_1, ..., f_p$ be semi-algebraic smooth functions on an open neighborhood of (x^0, y^0), such that $f_j(x^0, y^0) = 0$ for $j = 1, .., p$ and the matrix $[\frac{\partial f_j}{\partial y_i}(x^0, y^0)]$ is invertible. Then there exist open semi-algebraic neighborhoods U (resp. V) of x^0 (resp. y^0) in $\mathbb{R}^n$ (resp. $\mathbb{R}^p$) and a Nash mapping ϕ, such that $\phi(x^0) = y^0$ and $f_1(x, y) = ... = f_p(x, y) = 0 \Leftrightarrow y = \phi(x)$ for every $(x, y) \in U \times V$.*

Definition 12.2.3. *A* ***Nash submanifold*** *of $\mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ which is a smooth submanifold.*

By the implicit function theorem it is easy to see that this definition is equivalent to the following one, given in [BCR]:

Definition 12.2.4. *A semi-algebraic subset M of $\mathbb{R}^n$ is said to be a* ***Nash submanifold of*** $\mathbb{R}^n$ ***of dimension*** *d if, for every point x of M, there exists a Nash diffeomorphism ϕ from an open semi-algebraic neighborhood Ω of the origin in $\mathbb{R}^n$ onto an open semi-algebraic neighborhood Ω' of x in $\mathbb{R}^n$ such that $\phi(0) = x$ and $\phi(\mathbb{R}^d \times \{0\} \cap \Omega) = M \cap \Omega'$.*

Definition 12.2.5. *A* ***Nash map*** *from a Nash submanifold M of $\mathbb{R}^m$ to a Nash submanifold N of $\mathbb{R}^n$ is a semi-algebraic smooth map.*

Any open semi-algebraic subset of a Nash submanifold of $\mathbb{R}^n$ is also a Nash submanifold of $\mathbb{R}^n$.

Theorem 12.2.6 ([BCR, §2])**.** *Let $M \subset \mathbb{R}^n$ be a Nash submanifold. Then it has the same dimension as its Zarisky closure.*

Unfortunately, open semi-algebraic sets in $\mathbb{R}^n$ do not form a topology, but only a restricted topology. That is, the collection of open semi-algebraic sets is closed only under finite intersections and unions but not under infinite unions. For this reason we will consider only finite covers.
We will use this restricted topology to "glue" affine Nash manifolds and define Nash manifolds exactly in the same way as algebraic varieties are glued from affine algebraic varieties.

Definition 12.2.7. *A $\mathbb{R}$-**space** is a pair $(M, \mathcal{O}_M)$ where M is a restricted topological space and $\mathcal{O}_M$ a sheaf of $\mathbb{R}$-algebras over M which is a subsheaf of the sheaf $\mathbb{R}[M]$ of real-valued functions on M.*
*A **morphism between** $\mathbb{R}$-**spaces** $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ is a continuous map $f : M \to N$, such that the induced morphism of sheaves $f^* : f^*(\mathbb{R}[N]) \to \mathbb{R}[M]$ maps $\mathcal{O}_N$ to $\mathcal{O}_M$.*

Example 12.2.8. *Take for M a Nash submanifold of $\mathbb{R}^n$, and for $\mathring{\mathfrak{S}}(M)$ the family of all open subsets of M which are semi-algebraic in $\mathbb{R}^n$. For any open (semi-algebraic) subset U of M we take as $\mathcal{O}_M(U)$ the algebra $\mathcal{N}(U)$ of Nash functions $U \to \mathbb{R}$.*

Definition 12.2.9. *An **affine Nash manifold** is an $\mathbb{R}$-space which is isomorphic to an $\mathbb{R}$-space of a closed Nash submanifold of $\mathbb{R}^n$. A morphism between two affine Nash manifolds is a morphism of $\mathbb{R}$-spaces between them.*

Example 12.2.10. *Any real nonsingular affine algebraic variety has a natural structure of an affine Nash manifold.*

Remark 12.2.11. *Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be Nash submanifolds. Then a Nash map between them is the same as a morphism of affine Nash manifolds between them.*
Let $f : M \to N$ be a Nash map. Since an inverse of a semi-algebraic map is semi-algebraic, f is a diffeomorphism if and only if it is an isomorphism of affine Nash manifolds. Therefore we will call such f a Nash diffeomorphism.

In [Shi] there is another but equivalent definition of affine Nash manifold.

Definition 12.2.12. *An **affine** C^∞ **Nash manifold** is an $\mathbb{R}$-space which is isomorphic to an $\mathbb{R}$-space of a Nash submanifold of $\mathbb{R}^n$.*

The equivalence of the definitions follows from the following theorem, which immediately follows from [BCR, Theorem 8.4.6] and Proposition 12.1.16.

Theorem 12.2.13. *Any affine C^∞ Nash manifold is Nash diffeomorphic to a union of finite number of connected components of a real nonsingular affine algebraic variety.*

The book [Shi] usually uses the notion of affine C^ω Nash manifold instead of affine C^∞ Nash manifold, that we called here just Nash manifold. The two notions are equivalent by the theorem of Malgrange (see [Mal] or [Shi, Corollary I.5.7]) and hence equivalent to what we call just affine Nash manifold. In other words, any Nash manifold has a natural structure of a real analytic manifold and any Nash map between Nash manifolds is analytic.
One also considers C^r-Nash manifolds for any $0 \leq r < \infty$. These satisfy all the properties listed here, and in addition partition of unity.

Definition 12.2.14. *A **Nash manifold** is an $\mathbb{R}$-space $(M, \mathcal{N}_M)$ which has a finite cover (M_i) by open sets M_i such that the $\mathbb{R}$-spaces $(M_i, \mathcal{N}_M|_{M_i})$ are isomorphic to $\mathbb{R}$-spaces of affine Nash manifolds.*
*A **morphism between Nash manifolds** is a morphism of $\mathbb{R}$-spaces between them. Such morphisms are called Nash maps, and isomorphisms are called Nash diffeomorphisms.*

By Proposition 12.1.16, any Nash manifold is a union of a finite number of connected components. Any semi-algebraic set can be stratified by Nash manifolds.
Any Nash manifold has a natural structure of a smooth manifold. Any real nonsingular algebraic variety has a natural structure of a Nash manifold.
It is well-known that the real projective space $\mathbb{RP}^n$ is affine, see e.g. [BCR, Theorem 3.4.4]. Since any number of polynomial equations over $\mathbb{R}$ have the same set of solutions as a single equation (which is the sum of squares of the left hand sides), we get that any quasi-projective Nash manifold is affine.

Remark 12.2.15. *Note that the additive group of real numbers and and the multiplicative group of positive real numbers are isomorphic as Lie groups and as Nash manifolds, but are not isomorphic as Nash groups. Recently, the structure theory of (almost) linear Nash groups was developed in* [Sun].

The following theorem is a version of Hironaka's theorem for Nash manifolds.

Theorem 12.2.16 ([Shi, Corollary I.5.11])**.** *Let M be an affine Nash manifold. Then there exists a compact affine nonsingular algebraic variety N and a closed algebraic subvariety Z of N, which is empty if M is compact, such that Z has only normal crossings in N and M is Nash diffeomorphic to a union of connected components of $N - Z$.*

It implies that Nash manifolds are locally trivial.

Theorem 12.2.17 ([Shi, Theorem I.5.12])**.** *Any Nash manifold has a finite cover by open submanifolds Nash diffeomorphic to $\mathbb{R}^n$.*

Theorem 12.2.18 ([AG10, Theorem 2.4.3])**.** *Let M and N be Nash manifolds and $\nu : M \to N$ be a surjective submersive Nash map. Then locally (in the restricted topology) it has a Nash section, i.e. there exists a finite open cover $N = \bigcup_{i=1}^{k} U_i$ such that ν has a Nash section on each U_i.*

This implies that any etale Nash map is a local diffeomorphism.
In our work on Schwartz functions we frequently use the following

Theorem 12.2.19. *(Nash Tubular Neighborhood). Let $Z \subset M \subset \mathbb{R}^n$ be closed affine Nash submanifolds. Equip M with the Riemannian metric induced from $\mathbb{R}^n$. Then Z has a Nash tubular neighborhood, i.e. there exists a strictly positive Nash function $\rho_Z^M \in \mathcal{N}(Z)$ and a Nash diffeomorphism between between an open Nash neighborhood of Z in M and the open Nash neighborhood of the zero section of the normal bundle given by*

$$\{(z, v) \in N_Z^M \ s.t.\ ||v|| < \rho_Z^M(z).$$

12.3. **Schwartz functions on Nash manifolds.** The Fréchet space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions on $\mathbb{R}^n$ was defined by Laurant Schwartz to be the space of all smooth functions such that they and all their derivatives decay faster than $1/|x|^n$ for all n. In other words, $\mathcal{S}(\mathbb{R}^n)$ is the space of all $f \in C^\infty$ such that $|df|$ is bounded for every differential operator d with polynomial coefficients. This definition makes sense verbatim on any smooth affine algebraic variety and was extended in [dCl] to affine Nash manifolds, and in [AG08] (using some ideas from unpublished notes of Casselman) to arbitrary Nash manifolds. One can also define Schwartz sections of Nash bundles.

Definition 12.3.1. *Let M be a Nash manifold, and let E be a Nash bundle over it. Let $M = \bigcup_{i=1}^{k} U_i$ be an affine Nash trivialization of E. Then we have a map $\phi : \bigoplus_{i=1}^{k} \mathcal{S}(U_i)^n \to C^\infty(M, E)$. We define* ***the space $\mathcal{S}(M, E)$ of global Schwartz sections of*** *E by $\mathcal{S}(M, E) := Im\phi$. We define the topology on this space using the isomorphism $\mathcal{S}(M, E) \cong \bigoplus_{i=1}^{k} \mathcal{S}(U_i)^n / Ker\phi$.*

As Schwartz functions cannot be restricted to open subsets, but can be continued by 0 from open subsets, they form a cosheaf rather than a sheaf.
Let M be a Nash manifold, E be a Nash bundle over M and let $\mathcal{S}(M, E)$ denote the space of Schwartz sections of E.
The following two theorems summarize some results from [dCl, AG08, AG10, AG13].

Theorem 12.3.2. *Let $U \subset M$ be an open (Nash) submanifold and $N \subset M$ be a closed (Nash) submanifold. Then*

(1) $\mathcal{S}(\mathbb{R}^n)$ = Classical Schwartz functions on $\mathbb{R}^n$.

(2) $C_c^\infty(M,E) \subset \mathcal{S}(M,E)$.
(3) *The restriction maps* $\mathcal{S}(M,E)$ *onto* $\mathcal{S}(Z,E|_Z)$.
(4) $\mathcal{S}(U,E) := \mathcal{S}(U,E|_U) =$

$$\{\xi \in \mathcal{S}(M,E) | \xi \text{ vanishes with all its derivatives on } M-U\}.$$

(5) *Partition of unity: Let* $(U_i)_{i=1}^n$ *be a finite cover by open Nash submanifolds. Then there exist smooth functions* $\alpha_1,...,\alpha_n$ *such that* $supp(\alpha_i) \subset U_i$, $\sum_{i=1}^n \alpha_i = 1$ *and for any* $g \in \mathcal{S}(M,E)$, $\alpha_i g \in \mathcal{S}(U_i,E)$.
(6) $\mathcal{S}(M,E) = \mathcal{S}(M)\mathcal{S}(M,E)$.
(7) $\mathcal{S}(M,E)$ *is a nuclear Fréchet space.*
(8) *For any Nash manifold* M' *we have* $\mathcal{S}(M \times M') = \mathcal{S}(M)\hat{\otimes}\mathcal{S}(M')$.

Let us now sketch the proof of some parts of this theorem.
For part (1) we remark that Nash functions have polynomial growth.
Part (2) is obvious for affine M. For general M let $f \in C_c^\infty(M,E)$ and let $M = \bigcup_{i=1}^n U_i$ be an open affine cover of M. Choose a partition of unity $1 = \sum_i \alpha_i$ corresponding to this cover. Then $\alpha_i f \in C_c^\infty(U_i,E) \subset \mathcal{S}(U_i,E)$ and thus $f = \sum \alpha_i f \in \mathcal{S}(M,E)$.
Part (3) follows from the Nash tubular neighborhood theorem (Theorem 12.2.19).
Part (4) we will show later.
Part (5) is proven similarly to the classical partition of unity for smooth functions. We first prove “tempered" partition of unity, i.e. the existence of functions α_i supported in U_i and with polynomial growth, and then use this partition for a refined cover $V_i \subset U_i$. Then the temperedness of α_i guarantees $\alpha_i g \in \mathcal{S}(M)$ and supp$\alpha_i \subset V_i$ guarantees the vanishing of α_i near the boundary of U_i, which in turn implies $\alpha_i f \subset \mathcal{S}(U_i)$.
Part (6) we will not prove here due to lack of time.
For part (7) we note that a quotient of a nuclear Fréchet space by a closed subspace is nuclear Fréchet, and thus it is enough to show that $\mathcal{S}(\mathbb{R}^n)$ is nuclear Fréchet. This follows from its definition. For this we should say a couple of words on what nuclear means. It means that for any other Fréchet space, the projective and the injective topologies on their tensor product coincide. These are two natural topologies, and most other topologies are stronger than injective and weaker than projective. An inverse limit of spaces with inclusion maps that are Hilbert-Schmidt are nuclear, and that is why $\mathcal{S}(\mathbb{R}^n)$ is nuclear.
Part (8) we discussed already for compactly supported functions, and this discussion implies that $\mathcal{S}(M)\otimes\mathcal{S}(M')$ naturally embeds into $\mathcal{S}(M\times M')$ with dense image. We cannot prove the full statement at this point, since we have not discussed nuclear spaces and completed tensor products.
We denote by $\mathcal{S}^*(M)$ the dual space to $\mathcal{S}(M)$ and call it the space of tempered distributions. From property (4) we obtain

Corollary 12.3.3. *Let M be a Nash manifold and $U \subset M$ be an open Nash subset. Let E be a Nash bundle over M. Then we have a short exact sequence*

$$0 \to \mathcal{S}^*_{M\setminus U}(M) \to \mathcal{S}^*(M) \to \mathcal{S}^*(U) \to 0.$$

Theorem 12.3.4. *Let $N \subset M$ be a closed submanifold. Denote*

$$\mathcal{S}_N(M)^i := \{\phi \in \mathcal{S}(M) \text{ s.t. } \phi \text{ is } 0 \text{ on } N \text{ with first } i-1 \text{ derivatives}\}.$$

Let CN_M^N denote the conormal bundle to N in M. Then

$$\mathcal{S}(M)^i/\mathcal{S}(M)^{i+1} \cong \mathcal{S}(N, Sym^{i+1}(CN_M^N)) \tag{8}$$

$$\mathcal{S}(M \setminus N) \cong \lim_{\leftarrow} \mathcal{S}(M)^i/\mathcal{S}(M)^{i+1}. \tag{9}$$

Corollary 12.3.5. *Let M be a Nash manifold, and $N \subset M$ be a closed Nash submanifold. Then the space $\mathcal{S}^*_N(M)$ has a natural filtration F_i by the order of transversal derivatives. This filtration satisfies $\mathcal{S}^*_N(M) = \bigcup F_i$ and*

$$F_i/F_{i-1} \cong \mathcal{S}^*(N, Sym^{i+1}(CN_M^N)).$$

We frequently use this corollary when we have a natural stratification of M since it allows to reduce the analysis of equivariant distributions from M to single strata.

Theorem 12.3.6 ([AG09], Theorem B.2.4)**.** *Let $\phi : M \to N$ be a Nash submersion of Nash manifolds. Fix Nash measures μ on M and ν on N. Then*
(i) there exists a unique continuous linear map $\phi_* : \mathcal{S}(M) \to \mathcal{S}(N)$ such that for any $f \in \mathcal{S}(N)$ and $g \in \mathcal{S}(M)$ we have

$$\int_{x\in N} f(x)\phi_* g(x) d\nu = \int_{x\in M} (f(\phi(x)))g(x)d\mu.$$

In particular, we mean that both integrals converge.
(ii) If ϕ is surjective then ϕ_* is surjective.
In fact

$$\phi_* g(x) = \int_{z\in\phi^{-1}(x)} g(z)d\rho$$

for an appropriate measure ρ.

For the proof of part (4) we will need several lemmas.

Lemma 12.3.7. *Let $U \subset \mathbb{R}^n$ be an open semi-algebraic subset. For any $\phi : U \to \mathbb{R}$ denote by $\widetilde{\phi} : M \to \mathbb{R}$ its extension by 0 outside U. Let $\phi \in \mathcal{S}(U)$. Then $\widetilde{\phi}$ is differentiable at least once and for any Nash differential operator D of order 1 on $\mathbb{R}^n$, $D\widetilde{\phi} = \widetilde{D|_U \phi}$.*

Proof. We have to show that for any $z \in \mathbb{R}^n \setminus U$, $\widetilde{\phi}$ is differentiable at least once at z and its derivative at z in any direction is 0. Denote $F_z(x) := ||x - z||$. Clearly, $1/F_z^2 \in \mathcal{N}(U)$. Hence ϕ/F_z^2 is bounded in U and therefore $\widetilde{\phi}/F_z^2$ is bounded on $\mathbb{R}^n \setminus \{z\}$, which finishes the proof. ∎

Lemma 12.3.8 ([BCR, Proposition 2.6.4]). *Let $F : A \to \mathbb{R}$ be a semi-algebraic function on a locally closed semi-algebraic set. Let $Z(F) := \{x \in A | F(x) = 0\}$ be the set of zeros of F and let $A_F := A \setminus Z(F)$ be its complement. Let $G : A_F \to \mathbb{R}$ be a semi-algebraic function. Suppose that F and G are continuous. Then there exists an integer $N > 0$ such that the function $F^N G$, extended by 0 to $Z(F)$, is continuous on A.*

The following two lemmas are straightforward.

Lemma 12.3.9. *Let $U \subset \mathbb{R}^n$ be open (semi-algebraic) subset. Then any Nash differential operator D on U can be written as $\sum_{i=1}^k f_i(D_i|_U)$ where f_i are Nash functions on U and D_i are Nash differential operators on $\mathbb{R}^n$.*

Lemma 12.3.10. *Suppose $\alpha \in C^\infty(\mathbb{R})$ vanishes at 0 with all its derivatives. Then for any natural number n, $\alpha(t) = (n!)^{-1} t^n \alpha^{(n)}(\theta)$ for some $\theta \in [0, t]$.*

Proof of part (4). Denote $Z := X \setminus U$ and

$$W_Z := \{\xi \in \mathcal{S}(M, E) | \xi \text{ vanishes with all its derivatives on } M - U\}.$$

We have to show that the extension by zero defines a continuous isomorphism between $\mathcal{S}(U)$ and W_Z.

Case 1 $M = \mathbb{R}^N$.

Lemma 12.3.7 implies by induction that the extension by zero continuously maps $\mathcal{S}(U)$ into W_Z. Let us show that this map is onto.

Let $\phi \in W_Z$. For any point $x \in \mathbb{R}^N$ define $r(x) := dist(x, Z)$. Let $S := S(0, 1) \in \mathbb{R}^N$ be the unit sphere. Consider the function ψ on $S \times Z \times \mathbb{R}$ defined by $\psi(s, z, t) := \phi(z + ts)$. From Lemma 12.3.10 we see that

$$\psi(s, z, t) = t^n \frac{\partial^n}{(\partial t)^n} \psi(x, s, t)|_{t=\theta}$$

for some $\theta \in [0, t]$. As ϕ is Schwartz, it is easy to see that $\frac{\partial^n}{(\partial t)^n} \psi(x, s, t)$ is bounded on $Z \times S \times \mathbb{R}$. Therefore $|\psi(s, z, t)| \leq C|t|^n$ for some constant C and hence ϕ/r^n is bounded on $\mathbb{R}^N$ for any n.

Let h be a Nash function on U. By Lemma 12.3.8, $r^n h$ extends by 0 to a continuous semi-algebraic function on $\mathbb{R}^N$ for n big enough. It can be majorated by $f \in \mathcal{N}(\mathbb{R}^N)$. Therefore

$$|\phi h| = |(\phi/r^n) r^n h| \leq |\phi f|/r^n.$$

$\phi f \in W$, thus $|\phi f|/r^n$ is bounded and hence $|\phi h|$ is bounded.

For any Nash differential operator D on $\mathbb{R}^N$, $D\phi \in W$. Hence $hD\phi$ is bounded. By Lemma 12.3.9, every Nash differential operator on U is a sum of differential operators of the form $hD|_U$, where D is a Nash differential operator on $\mathbb{R}^N$ and h a Nash function on U. Hence $\phi|_U \in \mathcal{S}(U)$.

Case 2 M is affine.

Follows from the previous case and property (3) (extension from a closed Nash submanifold).

Case 3 General case.

Choose an affine cover of M. The theorem now follows from the previous case and partition of unity.

□

Recall that this property implies, by the Hahn-Banach theorem, that the restriction $\mathcal{S}^*(M) \to \mathcal{S}^*(U)$ is onto (Corollary 12.3.3). Let us demonstrate a classical corollary of this fact.

Corollary 12.3.11. *Any tempered generalized function on $\mathbb{R}^n$ has the form*

$$\sum_{i=1}^{k} \frac{\partial^{|\alpha_i|}}{(\partial x)^{\alpha_i}} f dx,$$

where α_i are multi-indexes, dx is a Lebesgue measure and f is a continuous function on $\mathbb{R}^n$, with $|f|$ bounded by a polynomial.

For example, $\delta_0 \in \mathcal{S}^*(\mathbb{R})$ is the second derivative of $|x|$.

Proof. Embed $\mathbb{R}$ into S^1. This defines an embedding of $\mathbb{R}^n$ into $T^n = (S^1)^n$. Let $\xi \in \mathcal{S}^*(\mathbb{R}^n)$. By Corollary 12.3.3, ξ extends to η on T^n. Let us use Fourier series. Distributions correspond under these to sequences indexed by n numbers that are bounded by some polynomial. Absolutely summable sequences define continuous functions. Partial derivatives of functions correspond to multiplications of sequences by polynomials in the indices. Thus, ξ corresponds to a sequence $\alpha_{i_1,\ldots,i_n}$ bounded by the product $i_1^{j_1} \cdots i_n^{j_n}$. Let $q(i_1, \ldots, i_n) := i_1^{j_1+2} \cdots i_n^{j_n+2}$ and let D be the corresponding differential operator with constant coefficients. Arguing by induction on n we can assume that if one of the numbers $i_1, \ldots, i_n$ vanishes then so does the corresponding coefficient $\alpha_{i_1,\ldots,i_n}$. Let $\beta := \alpha/q$ and let $h \in C(T^n)$ be the function with the Fourier series β. Then $\eta = D(hdt)$, where dt is the Haar probability measure on T^n. Thus $\xi = D|_{\mathbb{R}^n}(h|_{\mathbb{R}^n})$. Now, h is bounded as a function on T^n and thus $h|_{\mathbb{R}^n}$ is also bounded. When we take the stereographic projection into account we get that $D|_{\mathbb{R}^n}$ does not have constant coefficients. However, its coefficients are rational functions with nowhere vanishing denominator. Similarly, $dt|_{\mathbb{R}^n} = pdx$, where p is a rational function with nowhere vanishing denominator. Altogether, we get that there exists a differential operator Δ with constant coefficients and a continuous function f bounded by a polynomial such that

$$\xi = D|_{\mathbb{R}^n}(h|_{\mathbb{R}^n} dt|_{\mathbb{R}^n}) = \Delta f dx.$$

□

Corollary 12.3.12. *Let M be an affine Nash manifold and let ξ be a tempered generalized function on M. Then there exists a Nash differential operator D on M and a bounded continuous function $f \in C(M)$ such that $\xi = Df$.*

Proof. For the affine case embed M in $\mathbb{R}^n$ and extend ξ to $\mathbb{R}^n$. There find f and D from the previous corollary and restrict them to M. The general case follows by partition of unity. $\square$

Let us demonstrate how to use ?? on a simple example.

Lemma 12.3.13. *For any homogeneity degree α, the space of α-homogeneous even distributions on F is one-dimensional.*

Proof. By applying Fourier transform, we can assume $\alpha \geq 1$. To prove the non-vanishing, let ξ be $|x|^{\alpha-1}dx$.
To prove that any distribution in this space has to be a multiple of $|x|^{\alpha-1}dx$, it is enough to prove that the restriction to $F^\times$ is an embedding. Indeed, any distribution supported at 0 is a combination of derivatives of the δ_0 and thus has negative homogeneity degree. $\square$

The same statement with the same proof holds for odd distributions. Here, even distribution means invariant under the coordinate change $x \mapsto -x$, and odd means anti-invariant.
Finally, we would like to remark on a different, extrinsic, approach to Schwartz functions, applied in [CHM] and [KS]. We can compactify our manifold and define Schwartz functions on it as smooth functions on the (smooth) compactification that vanish to infinite order on the complement to M. If both M and the compactification are Nash manifolds then this definition will be equivalent, by Theorem 12.3.2(2,4). This allows to define Schwartz functions on non-Nash (say, subanalytic) manifolds, but this space will depend on the compactification.
Schwartz functions on non-smooth algebraic varieties and more generally on Nash varieties (i.e. varieties that can be locally described as zeros of Nash functions) can be locally defined by restriction from a smooth ambient space, see [ESh, Ela].
Another realm in which one can define Schwartz functions is the category of tempered manifolds. Indeed, our only use of (semi-)algebraicity was to have a scale of infinitesimals. Such a scale exists in the wider generality of tempered manifolds - manifolds that can be covered by open subsets that are identified with open subsets in $\mathbb{R}^n$, and coordinate changes between these subsets are tempered functions. A tempered manifold is said to be of finite type if this cover is finite. Many properties of Schwartz functions listed above continue to hold for tempered manifolds of finite type, see [Sha].

13. Invariant distributions

Let $\mathfrak{gl}_n(F)$ denote the vector space of all square matrices of order n with coefficients in F and $\mathrm{GL}_n(F) \subset \mathfrak{gl}_n(F)$ denote the group of all invertible matrices of order n. Consider also the embedding of $\mathrm{GL}_n(F)$ into $\mathrm{GL}_{n+1}(F)$ by $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$.

For a distribution $\xi \in \mathcal{S}^*(\mathrm{GL}_{n+1}(F))$ denote by ξ^t the distribution given by $\langle \xi^t, f\rangle := \langle \xi, f^t\rangle$, where $f^t(X) = f(X^t)$ and X^t denotes the transposed matrix. We let $\mathrm{GL}_n(F)$ act on $\mathrm{GL}_{n+1}(F)$ by conjugation via the embedding above. This action defines an action on Schwartz functions and thus also on tempered distributions. Denote by $\mathcal{S}^*(\mathrm{GL}_{n+1}(F))^{\mathrm{GL}_n(F)}$ the subspace of distributions invariant under this action. In this section we sketch the proof of the following theorem

Theorem 13.0.1. *For any $\xi \in \mathcal{S}^*(\mathrm{GL}_{n+1}(F))^{\mathrm{GL}_n(F)}$, we have $\xi = \xi^t$.*

13.1. **Proof for $n = 1$.** Note that $\mathrm{GL}_1(F) = F^\times$ and denote by $G := F^\times \ltimes \{\pm 1\}$, where -1 acts on $F^\times$ by $\lambda \mapsto \lambda^{-1}$. Extend the action of $F^\times$ on $\mathrm{GL}_2(F)$ to the action of G by letting $(1, -1)$ act by transposition. Let χ be the character of G given by projection on the second coordinate. Let $\mathcal{S}^*(\mathrm{GL}_2(F))^{G,\chi}$ denote the space of tempered distributions that change under the action of G by the character χ. Then the theorem is equivalent to the statement $\mathcal{S}^*(\mathrm{GL}_2(F))^{G,\chi} = 0$.

Proposition 13.1.1. *If $\mathcal{S}^*(\mathrm{gl}_2)^{G,\chi} = 0$ then $\mathcal{S}^*(\mathrm{GL}_2)^{G,\chi} = 0$.*

Proof. Let $\xi \in \mathcal{S}^*(\mathrm{GL}_2(F))^{G,\chi}$. We have to prove $\xi = 0$. Assume the contrary. Take $p \in \mathrm{Supp}(\xi)$. Let $t = \det(p)$. Let $f \in \mathcal{S}(F)$ be such that f vanishes in a neighborhood of zero and $f(t) \neq 0$. Consider the determinant map $\det : \mathrm{GL}_2(F) \to F$. Consider $\xi' := (f \circ \det) \cdot \xi$. It is easy to check that $\xi' \in \mathcal{S}^*(\mathrm{GL}_2(F))^{G,\chi}$ and $p \in \mathrm{Supp}(\xi')$. However, we can extend ξ' by zero to $\xi'' \in \mathcal{S}^*(\mathrm{gl}_2(F))^{G,\chi}$, which is zero by the assumption. Hence ξ' is also zero. Contradiction. $\square$

Now, note that the action of G on $\mathrm{gl}_2(F)$ is isomorphic to the action on $F^2 \times F^2$, where on the first copy the action is trivial and on the second copy it is given by

$$(\lambda, 1) \cdot (x, y) = (\lambda x, \lambda^{-1} y), \quad \text{and} \quad (1, -1) \cdot (x, y) = (y, x).$$

We can thus consider the second copy only, and prove $\mathcal{S}^*(F^2)^{G,\chi} = 0$.
Let $U := \{(x, y) \in F^2 \,|\, xy \neq 0\}$. Then U is locally isomorphic as a G-manifold to $F^\times \times F^\times$, where the action on the first $F^\times$ is trivial, and the action on the second one is given by $(\lambda, \varepsilon) \cdot z = (\lambda^2 z)^\varepsilon$. Clearly, with this action we have $\mathcal{S}^*(F^\times \times F^\times)^{G,\chi} = 0$ and thus we obtain

$$\mathcal{S}^*(U)^{G,\chi} = 0. \tag{10}$$

Now, denote by ρ the action of $F^\times$ on F^2 given by $\lambda(x, y) = (\lambda x, \lambda y)$. It defines an action on $\mathcal{S}(F^2)$ and $\mathcal{S}^*(F^2)$.

Lemma 13.1.2. *Let $\xi \in \mathcal{S}^*(F^2)^{G,\chi}$ and let $\sigma : F^\times \to \mathbb{C}$ be a character. Suppose that for any $\lambda \in F^\times$ and $f \in \mathcal{S}(F^2)$ we have $\langle \xi, \rho(\lambda^{-1})f\rangle = \sigma(\lambda)\langle \xi, f\rangle$. Then $|\sigma(\lambda)| = |\lambda|^{-n}$, where $n \geq 0$.*

Proof. Assume first that F is p-adic and consider the restriction of ξ to $F^\times \times F$. This restriction is supported on $F^\times \times \{0\}$ and thus is defined on $F^\times \times \{0\}$. Since it is $F^\times$-invariant, it is a multiple of the Haar measure. If it is a non-zero multiple, then $n = 0$. Now suppose that this restriction is 0. Similarly we get that either $n = 0$ or ξ is supported at the origin. In the latter case we again get $n = 0$.
Now, let F be $\mathbb{R}$ or $\mathbb{C}$ and consider the restriction of ξ to $F^\times \times F$, again supported on $F^\times \times \{0\}$. By ?? the space $\mathcal{S}^*(F^\times \times F)_{F^\times \times \{0\}}$ has a filtration by the order of transversal derivatives. The elements in the 0-th filtra are homogeneous as before. Since the derivation operators have negative homogeneity degrees, for other filtras we can have only negative degrees. □

Now, note that the Fourier transform preserves the space $\mathcal{S}^*(F^2)^{G,\chi}$. By (10), for any $\mathcal{S}^*(F^2)^{G,\chi} \subset \mathcal{S}^*_Z(F^2)$, where Z is the zero set of the quadratic form $q(x, y) = xy$. Thus, by Corollary 11.0.6, Proposition 11.0.9 and 13.1.2 we have $\mathcal{S}^*(F^2)^{G,\chi} = 0$.

13.2. **Luna slice theorem, Frobenius reciprocity and Harish-Chandra descent.** ??

13.3. **Sketch of Proof for all n.** ??

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