Recent applications of classical theorems on holonomic distributions

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Singapore, March 2016

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A $D$-module over a smooth affine algebraic variety $X$ is a module over the ring $D(X)$ of differential operators on $X$. A $D$-module $M$ given by generators and relations can be thought of as a system of PDE. A solution of $M$ is a $D$-module homomorphism of $M$ to an appropriate space of functions.

**Definition**

Let $M$ be a $D$-module over $X$ with generators $m_1 \ldots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree $\leq i$ and $F_i(M) := F_i(D(X))(m_1 \ldots m_k)$. Define

$$SS(M) := supp(gr_F(M)) \subset T^* X.$$ 

For a distribution $\xi$ on $X(\mathbb{R})$ define

$$SS(\xi) := SS(D(X)\xi) = \bigcap_{d\xi=0} Zeros(symbol(d)).$$

A $D$-module (or a distribution) $\xi$ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$
Theorem (Bernstein, cf. Sato)

(i) Holonomic D-modules have finite length.

(ii) Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a non-negative polynomial, and let $\xi \in S^* (\mathbb{R}^n)$ be a holonomic tempered distribution. Then the family of distributions $p^\lambda \xi$ defined for $\text{Re} \lambda > -1$ has a meromorphic continuation to $\lambda \in \mathbb{C}$.

This implies analytic continuation of Knap-Stein intertwining operators. Applied to symmetric pairs by van-den-Bahn, Brylinski, Delorme, Moelers-Osrted-Oshima,....

Corollary (Gourevitch-Sahi-Sayag)

Let a solvable real algebraic group $B$ act on an affine real algebraic manifold $X$. Let $\chi$ be a tempered character of $B$. Then $\dim S^* (X)^{B, \chi}$ is at least the number of open $B$-orbits in $X$ that possess $(B, \chi)$-equivariant measures.
Theorem (Bernstein, cf. Sato)

Let \( p \in \mathbb{R}[x_1, \ldots, x_n] \) be a non-negative polynomial, and let \( \xi \in S^*(\mathbb{R}^n) \) be a holonomic tempered distribution. Then the family of distributions \( p^\lambda \xi \) defined for \( \text{Re} \lambda > -1 \) has a meromorphic continuation to \( \lambda \in \mathbb{C} \).

Let a solvable real algebraic group \( B \) act on an affine real algebraic manifold \( X \), and let \( \mathcal{O} \) be an open orbit. Then \( \exists \) a \( B \)-equivariant polynomial \( p \neq 0 \) on \( X \) with \( p|_{X\setminus \mathcal{O}} = 0 \).

Corollary (Gourevitch-Sahi-Sayag)

\( \forall \) tempered \( \chi : B \rightarrow \mathbb{C}^\times \), if \( S^*(\mathcal{O})^B,\chi \neq 0 \) then \( S^*(X)^B,\chi \neq 0 \).

Proof.

For \( \xi \in S^*(\mathcal{O})^B,\chi \) and \( n \gg 0 \), \( p^n \xi \) extends to \( \eta \in S^*(X)^B,\psi^n\chi \). In the family \( p^\lambda \eta \), take \( \lambda = -n \). If this is a pole - take the principal part.

More generally, \( X \) can be quasiprojective and \( B \) can be MAN.
Let solvable $B$ act on affine $X$, and let $\mathcal{O}$ be an open orbit.

**Corollary (G.-Sahi-Sayag)**

$$\forall \text{ tempered } \chi : B \to \mathbb{C}^\times, \text{ if } S^\ast (\mathcal{O})^{B,\chi} \neq 0 \text{ then } S^\ast (X)^{B,\chi} \neq 0.$$  

**Proof.**

For $\xi \in S^\ast (\mathcal{O})^{B,\chi}$ and $n \gg 0$, $p^n \xi$ extends to $\eta \in S^\ast (X)^{B,\psi^n \chi}$. In the family $p^\lambda \eta$, take $\lambda = -n$. If this is a pole - take the principal part.

More generally, $X$ can be quasiprojective and $B$ can be MAN.

**Corollary (G.-Sahi-Sayag, in progress)**

Let $G$ be a real reductive group and $H \subset G$ be a real spherical subgroup. Let $C \subset P_0 \subset G$ be closed subgroup and let $V$ be tempered fin. dim. rep. of $C \times H$. Let $U \subset G$ be open $G$-invariant subset. Then $\dim S^\ast (G, V)^{C \times H} \geq \dim S^\ast (U, V)^{C \times H}$.

This gives Knap-Stein operators on spherical spaces and (degenerate) Whittaker models for (degenerate) principal series.
Theorem (Bernstein, cf. Kashiwara)

Let $X$ be a real algebraic manifold. Let $M$ be a holonomic right $D_X$-module. Then $\dim \text{Hom}(M, S^*(X)) < \infty$.

Theorem (Bernstein, Kashiwara, Aizenbud-G.-Minchenko)

Let $X$, $Y$ be smooth algebraic varieties and $\mathcal{M}$ be a family of holonomic $D_X$-modules parameterized by $Y$. Then $\dim \text{Hom}(\mathcal{M}_y, S^*(X))$ is bounded when $y$ ranges over $Y$.

Corollary (Aizenbud-G.-Minchenko 2015)

Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $\mathcal{E}$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a character of $\mathfrak{g}$. Then,

$$\dim S^*(X, \mathcal{E})^g,\chi < \infty.$$  

Moreover, it remains bounded when we change $\chi$ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.
Applications for co-invariants

Theorem (Aizenbud-G.-Krötz-Liu 2016)

Let a real algebraic group \( G \) act on a real algebraic manifold \( X \) with finitely many orbits. Let \( \mathcal{E} \) be an algebraic \( G \)-equivariant bundle on \( X \) and \( \chi \) be a tempered character of \( G \). Then,

\[
g(S(X, \mathcal{E}) \otimes \chi) \subset S(X, \mathcal{E}) \otimes \chi
\]

is closed and has finite codimension.

Corollary

Let \( G \) be a real reductive group, \( H \) be a real spherical subgroup, and \( \mathfrak{h} \) be the Lie algebra of \( H \). Let \( \chi \) be a tempered character of \( H \). Then for any admissible representation \( \pi \) of \( G \), \( H_0(\mathfrak{h}, \pi \otimes \chi) \) is separated and is non-degenerately paired with \( (\pi^*)^{\mathfrak{h},-\chi} \). In particular, the following conj. of Casselman are equivalent

- **Automatic continuity**: \( ((\pi^{HC})^*)^{\mathfrak{h}} \cong (\pi^*)^{\mathfrak{h}} \)
- **Comparison**: \( H_0(\mathfrak{h}, \pi^{HC}) \cong H_0(\mathfrak{h}, \pi) \)
We reprove the following theorem

**Theorem (Kobayashi-Oshima, Krötz-Schlichtkrull 2013)**

Let $G$ be a real reductive group, $H$ be a Zariski closed subgroup, and $\mathfrak{h}$ be the Lie algebra of $H$.

1. If $H$ is a spherical subgroup (i.e. $HB$ is open for some Borel subgroup $B$) then there exists $C \in \mathbb{N}$ such that $\dim(\pi^* \mathfrak{h}, \chi) \leq C$ for any $\pi \in \text{Irr}(G)$ and any character $\chi$ of $\mathfrak{h}$.

2. If $H$ is a real spherical subgroup (i.e. $HP$ is open for some minimal parabolic subgroup $P$) then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every $n$-dimensional representation $\tau$ of $\mathfrak{h}$ we have

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$
Sketch of the proof of Bernstein-Kashiwara theorem

- Enough to prove for the case $X$ is a vector space.
- Stone von-Neumann: The group $Sp(T^*(X))$ acts on the category of D-modules on $X$ stabilizing $S^*(X)$.
- $\dim SS_b = \dim SS_g$
- For $g \in Sp(T^*(X))$ we have, $g(SS_b(M)) = SS_b(gM)$
- $\exists g \in Sp(T^*(X))$ s.t. $g(SS_b(M)) \cap X^* = 0$
- This implies that $p : g(SS_b(M)) \to X$ is finite.
- This implies that $gM$ is smooth.
Theorem (Aizenbud-G.-Minchenko 2015)

Let $G$ be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups. The following system of equations on a distribution $\xi$ on $G$ is holonomic:

- $\xi$ is left $H_1$ invariant
- $\xi$ is right $H_2$ invariant
- $\xi$ is eigen w.r.t. the center $z(u(g))$ of the universal enveloping algebra of the Lie algebra of $G$. 
The spherical character

**Definition**

Let \((\pi, V)\) be an admissible representation of \(G(\mathbb{R})\) and \(v_1 \in (V^*)^{H_1}, \ v_2 \in (\tilde{V}^*)^{H_2}\). Define the spherical character of \(\pi\) w.r.t. \(v_1\) and \(v_2\) by:

\[
\langle \xi, f \rangle := \langle \pi^*(f)v_1, v_2 \rangle.
\]

**Corollary**

A spherical character of admissible representation w.r.t. pair of spherical groups is a holonomic distribution.

**Corollary (Aizenbud, Gourevitch, Minchenko, Sayag)**

For any local field \(F\), any spherical character of an admissible representation of \(G(F)\) is smooth in a Zariski open dense set.
**Theorem:** If $\#X/G < \infty$ then $gS(X) \subset S(X)$ is closed and has finite codimension.

**Proof:**

**Lemma (Aizenbud-Gourevitch-Krötz-Liu)**

$H_*(g, S(G/H))$ are finite dimensional (and Hausdorff).

Assume that $X = U \cup Z$ is a union of an open orbit and a closed one. It is enough to prove that $g(S(X)/S(Z)) \subset S(X)/S(Z)$ is closed and of finite co-dimension. Let $V := (S(X)/S(Z))$. The Borel’s lemma and the lemma above implies that $V$ is an inverse limit (with epimorphisms) of representations with finite dimensional co-homologies.

**Lemma**

*Such inverse limit commutes with homologies.*

On the other hand the Bernstein-Kashiwara theorem implies that $\dim(V^*)_g \leq S^*(X)_g < \infty$. 