Holonomicity of spherical characters and applications to multiplicity bounds

Dmitry Gourevitch

Weizmann Institute of Science

J. w. Avraham Aizenbud and Andrey Minchenko

http://www.wisdom.weizmann.ac.il/~dimagur/
A holonomic $D$-module over a smooth affine algebraic variety $X$ is a $D$-module over the ring $D(X)$ of differential operators on $X$. A $D$-module $M$ given by generators and relations can be thought of as a system of PDE. A solution of $M$ is a $D$-module homomorphism of $M$ to an appropriate space of functions.

**Definition**

Let $M$ be a $D$-module over $X$ with generators $m_1 \ldots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree $i$ and $F_i(M) := F_i(D(X))(m_1 \ldots m_k)$. Define $SS(M) := \text{supp}(\text{gr} F_i(M)) \subset T^*X$.

For a distribution $\xi$ on $X(\mathbb{R})$ define $SS(\xi) := SS(D(\mathbb{R}) \xi) = \bigcap_{d \xi = 0} \text{Zeros}(\text{symbol}(d))$.

A distribution (or a $D$-module) $\xi$ is called holonomic if $\dim(SS(\xi)) = \dim X$. 
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Let $M$ be a $D$-module over $X$ with generators $m_1, ..., m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree $i$ and $F_i(M) := F_i(D(X))(m_1, ..., m_k)$. Define $SS(M) := supp(\text{gr} F(M)) \subset T^*X$.

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Main results

Theorem (Aizenbud, G., Minchenko 2015)
Let G be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups (i.e. $H_i \backslash G / B$ is finite). The following system of equations on a distribution $\xi$ on G is holonomic:

- $\xi$ is left $H_1$ invariant
- $\xi$ is right $H_2$ invariant
- $\xi$ is eigen w.r.t. the center $z(u(g))$ of the universal enveloping algebra of the Lie algebra of G.

Corollary
Let $(\pi, V)$ be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)_{H_1}$, $v_2 \in (\tilde{V}^*)_{H_2}$. Let $\xi$ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f)v_1, v_2 \rangle.$$
Then $\xi$ is a holonomic distribution.
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Let $(\pi, V)$ be an admissible representation of $G$ over $\mathbb{R}$ and $v_1 \in (V^*)^{H_1}, v_2 \in (\tilde{V}^*)^{H_2}$. Let $\xi$ be the corresponding spherical character:

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Let $(\pi, V)$ be an admissible representation of $G(\mathbb{R})$ and $\nu_1 \in (V^*)^{H_1}$, $\nu_2 \in (\tilde{V}^*)^{H_2}$. Let $\xi$ be the corresponding spherical character:

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Then $\xi$ is a holonomic distribution.
Corollary (Aizenbud, G., Minchenko, Sayag)

Let $F$ be a local field of characteristic zero. Then the wave front set of any spherical character of an admissible representation of $G(F)$ is included in a conic subvariety of $T^*G$ of middle dimension. If $F = \mathbb{R}$ then the subvariety is Lagrangian.

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Any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.
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Theorem (Bernstein, Kashiwara ~1974)

Let $X$ be a real algebraic manifold. Let $M$ be a holonomic right $D_X$-module. Then $\dim \text{Hom}(M, S^*(X)) < \infty$. 

Corollary (Aizenbud, G., Minchenko)

Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $E$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a character of $g$. Then, $\dim S^*(X, E_g, \chi) < \infty$. Moreover, it remains bounded when we change $\chi$ or tensor $E$ with a representation of $g$ of a fixed dimension.
### Bernstein-Kashiwara Theorem

#### Theorem (Bernstein, Kashiwara ∼1974)

Let $X$ be a real algebraic manifold. Let $M$ be a holonomic right $\mathcal{D}_X$-module. Then $\dim \operatorname{Hom}(M, S^*(X)) < \infty$.

#### Theorem (Bernstein, Kashiwara, Aizenbud, Minchenko)

Let $X, Y$ be smooth algebraic varieties and $\mathcal{M}$ be a family of $\mathcal{D}_X$-modules parameterized by $Y$. Suppose that $\mathcal{M}_y$ is holonomic. Then $\dim \operatorname{Hom}(\mathcal{M}_y, S^*(X))$ is bounded when $y$ ranges over $Y$.

### Corollary (Aizenbud, Minchenko)

Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $E$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a character of $g$. Then, $\dim S^*(X, E^g, \chi) < \infty$. Moreover, it remains bounded when we change $\chi$ or tensor $E$ with a representation of $g$ of a fixed dimension.
### Bernstein-Kashiwara theorem

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Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $\mathcal{E}$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a character of $\mathfrak{g}$. Then,

$$\dim S^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.$$ 

Moreover, it remains bounded when we change $\chi$ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.
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**Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)**

*Let $G$ be a real reductive group, $H$ be a Zariski closed subgroup, and $\mathfrak{h}$ be the Lie algebra of $H$. If $H$ is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^* \mathfrak{h}, \chi) \leq C$ for any $\pi \in \text{Irr}(G)$ and any character $\chi$ of $\mathfrak{h}$. If $H$ is a real spherical subgroup then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every $n$-dimensional representation $\tau$ of $\mathfrak{h}$ we have $\dim \text{Hom}_\mathfrak{h}(\pi, \tau) \leq C_n$.***
Applications to multiplicities

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Theorem (Aizenbud, G., Krötz, Liu)

Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $\mathcal{E}$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a tempered character of $G$. Then, the homology $H_0(g, S(X, \mathcal{E}) \otimes \chi)$ is separated and is non-degenerately paired with $S^*(X, \mathcal{E})^{g, -\chi}$. I.e.

$$gS(X, \mathcal{E}) \otimes \chi \subset S(X, \mathcal{E}) \otimes \chi$$

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$$\mathfrak{g}S(X, \mathcal{E}) \otimes \chi \subset S(X, \mathcal{E}) \otimes \chi$$

is closed and has finite codimension.

Corollary

Let $G$ be a real reductive group, $H$ be a real spherical subgroup, and $\mathfrak{h}$ be the Lie algebra of $H$. Let $\chi$ be a tempered character of $H$. Then for any admissible representation $\pi$ of $G$, $H_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^*)^\mathfrak{h}, \chi$. 
Theorem (A., Gourevitch, Minchenko 2015)

Let

\[ S = \{ g \in G, x \in g^* | x \in \mathfrak{h}_1^\perp, \text{ad}(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent} \} = G \times \mathcal{N} \cap \bigcup_{g \in G} \mathcal{CN}_G^{\mathcal{H}_1}g \mathcal{H}_2, g \]

Then

\[ \dim S = \dim G . \]
Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$. 
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So

$$S' \subset \bigcup_{x \in \mathcal{N}_H} \mathcal{C}N^h_{\text{ad}(G)x,x}$$
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Theorem (Steinberg 1976) \( \forall \eta \in N \) we have \[ \dim \mathfrak{g}_\eta - 2 \dim \mu - 1 (\eta) = \mathrm{rk} \mathfrak{g}. \]
Let $\mathcal{B}$ be the flag variety. $T^*\mathcal{B} \cong \{ B \in \mathcal{B}, x \in \mathfrak{b}^\perp \}$. We have a natural map $\mu : T^*\mathcal{B} \to N$. It is called the Springer resolution.

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**Theorem (Steinberg 1976)**

$\forall \eta \in \mathcal{N}$ we have $\dim G_\eta - 2 \dim \mu^{-1}(\eta) = \text{rk} G$. 
Idea of the proof

\[ T^*B \times T^*B \times \mu \times \mu \to G \times N \]
\[ (g, x) \mapsto (x, \text{Ad}^{-1}(g)x) \]

Passing to the fiber of \( 0 \in h^*1 \times h^*2 \) we get:
\[ L^1 \times L^2 \to S \]
\[ N^*h_1 \times N^*h_2 \]

Where \( N^*h_i = N \cap h_i^\perp \) and \( L_i = \{ (B, X) \in T^*B | X \in h_i^\perp \} = \bigcup_{x \in B} C_{N^*B} H_i x, x \).

The estimate on \( \dim S \) follows from the Steinberg theorem and:
\[ \dim L_i = \dim B \]
Idea of the proof

\[ T^*B \times T^*B \quad \begin{array}{c} \mu \times \mu \\
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\[ (g,x) \mapsto (x, \text{Ad}^*(g^{-1})x) \]

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\[ L^1 \times L^2 \rightarrow S \rightarrow N \quad \mathcal{h}_1^* \times \mathcal{h}_2^* \]

Where \( \mathcal{N}_h_i := \mathcal{N} \cap h_i^\perp \) and \( L_i := \{ (B, X) \in T^*B | X \in h_i^\perp \} = \bigcup_{x \in B} \mathbb{C}N_B H_i x, x \).

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\[ \mathcal{N}_{h_1} \times \mathcal{N}_{h_2} \rightarrow S \]

Where

\[ \mathcal{N}_{h_i} := \mathcal{N} \cap \mathfrak{h}_i^\perp \text{ and } L_i := \{(B, X) \in T^*\mathcal{B} \mid X \in \mathfrak{h}_i^\perp\} \]
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\[ T^*B \times T^*B \xrightarrow{\mu \times \mu} G \times N \xrightarrow{(g,x) \mapsto (x, \text{Ad}^* (g^{-1}) x)} \]

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