Holonomicity of spherical characters and applications to multiplicity bounds

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Holonomic D-modules and distributions

A $D$-module over a smooth affine algebraic variety $X$ is a module over the ring $D(X)$ of differential operators on $X$. A $D$-module $M$ given by generators and relations can be thought of as a system of PDE. A solution of $M$ is a $D$-module homomorphism of $M$ to an appropriate space of functions.

**Definition**

Let $M$ be a $D$-module over $X$ with generators $m_1 \ldots m_k$. Define $F_i(D(X))$ to be the space of differential operators of degree $i$ and $F_i(M) := F_i(D(X))(m_1 \ldots m_k)$. Define

$$SS(M) := \text{supp}(\text{gr}_F(M)) \subset T^*X.$$ 

For a distribution $\xi$ on $X(\mathbb{R})$ define

$$SS(\xi) := SS(D(X)\xi) = \bigcap_{d\xi=0} \text{Zeros}(\text{symbol}(d)).$$

A distribution (or a $D$-module) $\xi$ is called holonomic if

$$\dim(SS(\xi)) = \dim X.$$
Main results

**Theorem (Aizenbud, G., Minchenko 2015)**

Let $G$ be an algebraic reductive group, $H_1, H_2 \subset G$ be spherical subgroups (i.e. $H_i \backslash G/B$ is finite). The following system of equations on a distribution $\xi$ on $G$ is holonomic:

- $\xi$ is left $H_1$ invariant
- $\xi$ is right $H_2$ invariant
- $\xi$ is eigen w.r.t. the center $\mathfrak{z}(u(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra of $G$.

**Corollary**

Let $(\pi, V)$ be an admissible representation of $G(\mathbb{R})$ and $v_1 \in (V^*)^{H_1}$, $v_2 \in (\tilde{V}^*)^{H_2}$. Let $\xi$ be the corresponding spherical character:

$$\langle \xi, f \rangle := \langle \pi^*(f)v_1, v_2 \rangle.$$

Then $\xi$ is a holonomic distribution.
Corollary (Aizenbud, G., Minchenko, Sayag)

Let $F$ be a local field of characteristic zero. Then the wave front set of any spherical character of an admissible representation of $G(F)$ is included in a conic subvariety of $T^* G$ of middle dimension. If $F = \mathbb{R}$ then the subvariety is Lagrangian.

Corollary

Any spherical character of an admissible representation of $G(F)$ is smooth in a Zariski open dense set.
Bernstein-Kashiwara theorem

**Theorem (Bernstein, Kashiwara ~1974)**

Let $X$ be a real algebraic manifold. Let $M$ be a holonomic right $D_X$-module. Then \( \dim \text{Hom}(M, S^*(X)) < \infty \).

**Theorem (Bernstein, Kashiwara, Aizenbud, G., Minchenko)**

Let $X, Y$ be smooth algebraic varieties and $M$ be a family of $D_X$-modules parameterized by $Y$. Suppose that $M_y$ is holonomic. Then \( \dim \text{Hom}(M_y, S^*(X)) \) is bounded when $y$ ranges over $Y$.

**Corollary (Aizenbud, G., Minchenko)**

Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $\mathcal{E}$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a character of $\mathfrak{g}$. Then,

\[
\dim S^*(X, \mathcal{E})^{\mathfrak{g}, \chi} < \infty.
\]

Moreover, it remains bounded when we change $\chi$ or tensor $\mathcal{E}$ with a representation of $\mathfrak{g}$ of a fixed dimension.
We reprove the following theorem

**Theorem (Kobayashi, Krötz, Oshima, Schlichtkrull 2013)**

Let $G$ be a real reductive group, $H$ be a Zariski closed subgroup, and $\mathfrak{h}$ be the Lie algebra of $H$.

1. If $H$ is a spherical subgroup then there exists $C \in \mathbb{N}$ such that $\dim(\pi^* \mathfrak{h}, \chi) \leq C$ for any $\pi \in \text{Irr}(G)$ and any character $\chi$ of $\mathfrak{h}$.

2. If $H$ is a real spherical subgroup then, for every irreducible admissible representation $\pi \in \text{Irr}(G)$, and natural number $n \in \mathbb{N}$ there exists $C_n \in \mathbb{N}$ such that for every $n$-dimensional representation $\tau$ of $\mathfrak{h}$ we have

$$\dim \text{Hom}_{\mathfrak{h}}(\pi, \tau) \leq C_n.$$
Corollaries for homologies

Theorem (Aizenbud, G., Krötz, Liu)

Let a real algebraic group $G$ act on a real algebraic manifold $X$ with finitely many orbits. Let $\mathcal{E}$ be an algebraic $G$-equivariant bundle on $X$ and $\chi$ be a tempered character of $G$. Then, the homology $H_0(g, S(X, \mathcal{E}) \otimes \chi)$ is separated and is non-degenerately paired with $S^*(X, \mathcal{E})^g, -\chi$. I.e.

$$gS(X, \mathcal{E}) \otimes \chi \subset S(X, \mathcal{E}) \otimes \chi$$

is closed and has finite codimension.

Corollary

Let $G$ be a real reductive group, $H$ be a real spherical subgroup, and $\mathfrak{h}$ be the Lie algebra of $H$. Let $\chi$ be a tempered character of $H$. Then for any admissible representation $\pi$ of $G$, $H_0(\mathfrak{h}, \pi \otimes \chi)$ is separated and is non-degenerately paired with $(\pi^*)^\mathfrak{h}, -\chi$. 
Theorem (A., Gourevitch, Minchenko 2015)

Let

\[ S = \{ g \in G, x \in g^* \mid x \in \mathfrak{h}_1^\perp, \text{ad}(g)(x) \in \mathfrak{h}_2^\perp, x \text{ is nilpotent} \} = \]

\[ = G \times \mathcal{N} \cap \bigcup_{g \in G} \text{CN}_{H_1gH_2,g}^G \]

Then

\[ \dim S = \dim G. \]
Assume $H_1 = H_2 = H$, diagonally embedded in $G = H \times H$. Translating the problem to $H = G/H$ we obtain:

\[ S' = \{ g \in H, x \in \mathfrak{h}^* | \text{Ad}(g)(x) = x, x \in \mathcal{N}_H \} = H \times \mathcal{N}_H \cap \bigcup_{g \in H} \mathcal{C}N_{\text{ad}(G)g,g}^H \]

passing to the Lie algebra

\[ S' = \{ g \in \mathfrak{h}, x \in \mathfrak{h} | [x, g] = 0, x \text{ is nilpotent} \} \]

So

\[ S' \subset \bigcup_{x \in \mathcal{N}_H} \mathcal{C}N_{\text{ad}(G)x,x}^\mathfrak{h} \]
Let $\mathcal{B}$ be the flag variety. $T^*\mathcal{B} \cong \{B \in \mathcal{B}, x \in b^\perp\}$. We have a natural map $\mu : T^*\mathcal{B} \to \mathcal{N}$. It is called the Springer resolution.

Theorem (Steinberg 1976)

\[ \forall \eta \in \mathcal{N} \ we \ have \ \dim G_\eta - 2 \dim \mu^{-1}(\eta) = \text{rk} G. \]
Idea of the proof

\[ T^*B \times T^*B \xrightarrow{\mu \times \mu} G \times N \]

\[ (g, x) \mapsto (x, \text{Ad}^*(g^{-1})x) \]

\[ N \times N \xrightarrow{\text{res}} h^*_1 \times h^*_2 \]

Passing to the fiber of \(0 \in h^*_1 \times h^*_2\) we get:

\[ L_1 \times L_2 \]

Where

\[ N_{h_i} := N \cap h_i^\perp\]

and

\[ L_i := \left\{ (B, X) \in T^*B \mid X \in h_i^\perp \right\} = \bigcup_{x \in B} CN_{h_i | x, x}^B \]

The estimate on \(\dim S\) follows from the Steinberg theorem and:

\[ \dim L_i = \dim B. \]