Multiplicity One Theorems

D. Gourevitch

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Let $F$ be a local field of characteristic zero.

**Theorem (Aizenbud-Gourevitch-Rallis-Schiffmann-Sun-Zhu)**

Every $GL_n(F)$-invariant distribution on $GL_{n+1}(F)$ is transposition invariant.
Formulation

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It has the following corollary in representation theory.

**Theorem**

*Let $\pi$ be an irreducible admissible representation of $GL_{n+1}(F)$ and $\tau$ be an irreducible admissible representation of $GL_n(F)$. Then*

\[ \dim \text{Hom}_{GL_n(F)}(\pi, \tau) \leq 1. \]
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$$\dim \text{Hom}_{GL_n(F)}(\pi, \tau) \leq 1.$$  

*Similar theorems hold for orthogonal and unitary groups.*
Let $M$ be a smooth manifold. We denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on $M$. We will consider the space $(C_c^\infty(M))^*$ of distributions on $M$. Sometimes we will also consider the space $S^*(M)$ of Schwartz distributions on $M$. 
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Definition

An $\ell$-space is a Hausdorff locally compact totally disconnected topological space. For an $\ell$-space $X$ we denote by $S(X)$ the space of compactly supported locally constant functions on $X$. We let $S^*(X) := S(X)^*$ be the space of distributions on $X$. 
\[ \tilde{G} := GL_n(F) \rtimes \{1, \sigma\} \]

Define a character \( \chi \) of \( \tilde{G} \) by \( \chi(GL_n(F)) = \{1\} \),\
\( \chi(\tilde{G} - GL_n(F)) = \{-1\} \).
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Equivalent formulation:

**Theorem**

\[ S^*(GL_{n+1}(F))^{\tilde{G},\chi} = 0. \]
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\[ S^* (gl_{n+1}(F)) \tilde{G},\chi = 0. \]

\[
g \left( \begin{pmatrix} A_{n \times n} & v_{n \times 1} \\ \phi_{1 \times n} & \lambda \end{pmatrix} \right) g^{-1} = \begin{pmatrix} gAg^{-1} & gv \\ (g^*)^{-1} & g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & v \\ \phi & \lambda \end{pmatrix}^t = \begin{pmatrix} A^t & \phi^t \\ v^t & \lambda \end{pmatrix} \]
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Equivalent formulation:

**Theorem**

\[ S^*(X)^{\tilde{G},\chi} = 0. \]
A group $G$ acts on a space $X$, and $\chi$ is a character of $G$. We want to show $S^*(X)^G,\chi = 0$. 

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Proposition

Let $U \subset X$ be an open $G$-invariant subset and $Z := X - U$. Suppose that $S^*(U)^G,\chi = 0$ and $S^*_X(Z)^G,\chi = 0$. Then $S^*(X)^G,\chi = 0$. 
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**Proposition**

Let $U \subset X$ be an open $G$-invariant subset and $Z := X - U$. Suppose that $S^*(U)^G,\chi = 0$ and $S^*_\chi(Z)^G,\chi = 0$. Then $S^*(X)^G,\chi = 0$.

**Proof.**

$$0 \to S^*_\chi(Z)^G,\chi \to S^*(X)^G,\chi \to S^*(U)^G,\chi.$$
First tool: Stratification

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For $\ell$-spaces, $S^*_\chi(Z)^G,\chi \cong S^*(Z)^G,\chi$. 
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For smooth manifolds, there is a slightly more complicated statement which takes into account transversal derivatives.
Theorem (Bernstein, Baruch, ...)

Let \( \psi : X \to Z \) be a map.
Let \( G \) act on \( X \) and \( Z \) such that \( \psi(gx) = g\psi(x) \).
Suppose that the action of \( G \) on \( Z \) is transitive.
Suppose that both \( G \) and \( \text{Stab}_G(z) \) are unimodular. Then

\[
S^*(X)^{G,\chi} \cong S^*(X_z)^{\text{Stab}_G(z),\chi}.
\]
Generalized Harish-Chandra descent

Theorem

Let a reductive group $G$ act on a smooth affine algebraic variety $X$. Let $\chi$ be a character of $G$. Suppose that for any $a \in X$ s.t. the orbit $Ga$ is closed we have

$$S^*(N^X_{Ga,a})^{Ga,\chi} = 0.$$ 

Then $S^*(X)^{G,\chi} = 0$. 

D. Gourevitch

Multiplicity One Theorems
Let $V$ be a finite dimensional vector space over $F$ and $Q$ be a non-degenerate quadratic form on $V$. Let $\widehat{\xi}$ denote the Fourier transform of $\xi$ defined using $Q$.

**Proposition**

Let $G$ act on $V$ linearly and preserving $Q$. Let $\xi \in S^*(V)^G,\chi$. Then $\widehat{\xi} \in S^*(V)^G,\chi$. 

D. Gourevitch

Multiplicity One Theorems
We call a distribution $\xi \in S^*(V)$ **abs-homogeneous of degree** $d$ if for any $t \in F^\times$, 

$$h_t(\xi) = u(t)|t|^d \xi,$$

where $h_t$ denotes the homothety action on distributions and $u$ is some unitary character of $F^\times$. 

**Theorem (Jacquet, Rallis, Schiffmann,...)**

Assume $F$ is non-archimedean. Let $\xi \in S^*(V)$ be such that $\hat{\xi} \in S^*(V)$. Then $\xi$ is abs-homogeneous of degree $\frac{1}{2} \dim V$.

**Theorem (archimedean homogeneity)**

Let $F$ be any local field. Let $L \subset S^*(V)$ be a non-zero linear subspace such that $\forall \xi \in L$ we have $\hat{\xi} \in L$ and $Q\xi \in L$. Then there exists a non-zero distribution $\xi \in L$ which is abs-homogeneous of degree $\frac{1}{2} \dim V$ or of degree $\frac{1}{2} \dim V + 1$. 
We call a distribution $\xi \in \mathcal{S}^*(V)$ **abs-homogeneous of degree** $d$ if for any $t \in F^\times$,

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**Theorem (Jacquet, Rallis, Schiffmann,...)**

*Assume $F$ is non-archimedean.* Let $\xi \in \mathcal{S}^*_V(Z(Q))$ be s.t. $\hat{\xi} \in \mathcal{S}^*_V(Z(Q))$. Then $\xi$ is abs-homogeneous of degree $\frac{1}{2} \dim V$. 
Fourier transform and homogeneity

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To a distribution $\xi$ on $X$ one assigns two subsets of $T^*X$. 
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In the non-Archimedean case we define the singular support to be the Zariski closure of the wave front set.
Let $X$ be a smooth algebraic variety.

- Let $\xi \in S^*(X)$. Then $\text{Supp}(\xi)_{\text{Zar}} = p_X(\text{SS}(\xi))$, where $p_X : T^*X \to X$ is the projection.
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- Let $\xi \in S^*(X)$. Then $\text{Supp}(\xi)_{\text{Zar}} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.
- Let an algebraic group $G$ act on $X$. Let $\xi \in S^*(X)^G,\chi$. Then

$$SS(\xi) \subset \{(x, \phi) \in T^*X \mid \forall \alpha \in g \quad \phi(\alpha(x)) = 0\}.$$
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- Let $\xi \in S^*(X)$. Then $\text{Supp}(\xi)_{\text{Zar}} = p_X(SS(\xi))$, where $p_X : T^*X \to X$ is the projection.
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  \[ SS(\xi) \subset \{ (x, \phi) \in T^*X \mid \forall \alpha \in g \; \phi(\alpha(x)) = 0 \} . \]

- Let $V$ be a linear space. Let $Z \subset V^*$ be a closed subvariety, invariant with respect to homotheties. Let $\xi \in S^*(V)$. Suppose that $\text{Supp}(\hat{\xi}) \subset Z$. Then $SS(\xi) \subset V \times Z$. 

**Integrability Theorem:**

Let $\xi \in S^*(X)$. Then $SS(\xi)$ is (weakly) coisotropic.
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- Integrability theorem:
  Let $\xi \in S^*(X)$. Then $\text{SS}(\xi)$ is (weakly) coisotropic.
Coisotropic varieties

**Definition**

Let $M$ be a smooth algebraic variety and $\omega$ be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it $M$-coisotropic if the following equivalent conditions hold.

- At every smooth point $z \in Z$ we have $T_z Z \supset (T_z Z)^\perp$. Here, $(T_z Z)^\perp$ denotes the orthogonal complement to $T_z Z$ in $T_z M$ with respect to $\omega$.
- The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.

If there is no ambiguity, we will call $Z$ a coisotropic variety.
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If there is no ambiguity, we will call $Z$ a coisotropic variety.

- Every non-empty coisotropic subvariety of $M$ has dimension at least $\frac{\dim M}{2}$.
Weakly coisotropic varieties

Definition

Let $X$ be a smooth algebraic variety. Let $Z \subset T^*X$ be an algebraic subvariety. We call it $T^*X$-weakly coisotropic if one of the following equivalent conditions holds.

- For a generic smooth point $a \in p_X(Z)$ and for a generic smooth point $y \in p_X^{-1}(a) \cap Z$ we have $CN^X_{p_X(Z),a} \subset T_y(p_X^{-1}(a) \cap Z)$.
- For any smooth point $a \in p_X(Z)$ the fiber $p_X^{-1}(a) \cap Z$ is locally invariant with respect to shifts by $CN^X_{p_X(Z),a}$.

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Multiplicity One Theorems
Definition

Let $X$ be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety and $R \subset T^*X$ be any subvariety. We define the restriction $R|_Z \subset T^*Z$ of $R$ to $Z$ by

$$R|_Z := q(p_X^{-1}(Z) \cap R),$$

where $q : p_X^{-1}(Z) \to T^*Z$ is the projection.

$$T^*X \supset p_X^{-1}(Z) \to T^*Z$$
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Lemma

Let $X$ be a smooth algebraic variety. Let $Z \subset X$ be a smooth subvariety. Let $R \subset T^*X$ be a (weakly) coisotropic variety. Then, under some transversality assumption, $R|_Z \subset T^*Z$ is a (weakly) coisotropic variety.
Harish-Chandra descent and homogeneity

Notation

\[ S := \{(A, v, \phi) \in X_n | A^n = 0 \text{ and } \phi(A^i v) = 0 \forall 0 \leq i \leq n\}. \]
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By Harish-Chandra descent we can assume that any \( \xi \in S^*(X)^{\tilde{G},\chi} \) is supported in \( S \).
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\[ S' := \{(A, \nu, \phi) \in S | A^{n-1}\nu = (A^*)^{n-1}\phi = 0 \}. \]
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By the homogeneity theorem, the stratification method and Frobenius descent we get that any \( \xi \in S^*(X)^\tilde{G},\chi \) is supported in \( S' \).
Reduction to the geometric statement

Notation

\[ T' = \left\{ ((A_1, v_1, \phi_1), (A_2, v_2, \phi_2)) \in X \times X \mid \forall i, j \in \{1, 2\} \right\} \]

\[ \text{and} \left\{ (A_i, v_j, \phi_j) \in S' \text{ and } [A_1, A_2] + v_1 \otimes \phi_2 - v_2 \otimes \phi_1 = 0 \right\}. \]
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It is enough to show:

Theorem (The geometric statement)

There are no non-empty \( X \times X \)-weakly coisotropic subvarieties of \( T' \).
Reduction to the Key Lemma

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\[ T'' := \{(A_1, v_1, \phi_1), (A_2, v_2, \phi_2) \in T' \mid A_1^{n-1} = 0\}. \]
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**Notation**

Let \( A \in \mathfrak{sl}(V) \) be a nilpotent Jordan block. Denote

\[ R_A := (T' - T'')|_{\{A\} \times V \times V^*}. \]
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Lemma (Key Lemma)

There are no non-empty \( V \times V^* \times V \times V^* \)-weakly coisotropic subvarieties of \( R_A \).
Proof of the Key Lemma

Notation

\[ Q_A := S' \cap (\{A\} \times V \times V^*) = \bigcup_{i=1}^{n-1} (\text{Ker}A^i) \times (\text{Ker}(A^*)^{n-i}) \]

It is easy to see that \( R_A \subset Q_A \times Q_A \) and \( Q_A \times Q_A = \bigcup_{i,j=1}^{n-1} L_{ij} \), where

\[ L_{ij} := (\text{Ker}A^i) \times (\text{Ker}(A^*)^{n-i}) \times (\text{Ker}A^j) \times (\text{Ker}(A^*)^{n-j}) \]

It is easy to see that any weakly coisotropic subvariety of \( Q_A \times Q_A \) is contained in \( \bigcup_{i=1}^{n-1} L_{ii} \).

Hence it is enough to show that for any \( 0 < i < n \), we have \( \dim(R_A \cap L_{ii}) < 2^{n-1} \).

Let \( f \in O(L_{ii}) \) be the polynomial defined by

\[ f(v_1, \phi_1, v_2, \phi_2) := (v_1)^i(\phi_2)^i+1 - (v_2)^i(\phi_1)^i+1. \]

It is enough to show that \( f(R_A \cap L_{ii}) = \{0\} \).
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\[ Q_A := S' \cap (\{A\} \times V \times V^*) = \bigcup_{i=1}^{n-1} (\text{Ker}A^i) \times (\text{Ker}(A^*)^{n-i}) \]

It is easy to see that \( R_A \subset Q_A \times Q_A \) and \( Q_A \times Q_A = \bigcup_{i,j=1}^{n-1} L_{ij} \), where

\[ L_{ij} := (\text{Ker}A^i) \times (\text{Ker}(A^*)^{n-i}) \times (\text{Ker}A^j) \times (\text{Ker}(A^*)^{n-j}). \]

It is easy to see that any weakly coisotropic subvariety of \( Q_A \times Q_A \) is contained in \( \bigcup_{i=1}^{n-1} L_{ii} \). Hence it is enough to show that for any \( 0 < i < n \), we have \( \dim R_A \cap L_{ii} < 2n \).
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\[ f(v_1, \phi_1, v_2, \phi_2) := (v_1)_i(\phi_2)_{i+1} - (v_2)_i(\phi_1)_{i+1}. \]
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It is enough to show that \( f(R_A \cap L_{ii}) = \{0\} \).
Let \((v_1, \phi_1, v_2, \phi_2) \in L_{ii}\). Let \(M := v_1 \otimes \phi_2 - v_2 \otimes \phi_1\).
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M = \begin{pmatrix}
0_{i \times i} & * \\
0_{(n-i) \times i} & 0_{(n-i) \times (n-i)}
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We know that there exists a nilpotent \(B\) satisfying \([A, B] = M\). Hence this \(B\) is upper nilpotent, which implies \(M_{i, i+1} = 0\) and hence \(f(v_1, \phi_1, v_2, \phi_2) = 0\).
$sl(V) \times V \times V^*$
\[ sl(V) \times V \times V^* \rightarrow_{H.Ch. descent} S \]
Flowchart

\[ sl(V) \times V \times V^* \xrightarrow{H. Ch. descent} S \xrightarrow{\text{Fourier transform and homogeneity theorem}} S' \]
Summary

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$sl(V) \times V \times V^* \xrightarrow{\text{H.Ch. descent}} S \xrightarrow{\text{Fourier transform and homogeneity theorem}} S' \xrightarrow{\text{Fourier transform and integrability theorem}} T'$
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\[ T' - T'' \]
Summary

Flowchart

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\[ R_A \xleftarrow{\text{restriction}} T' \rightarrow T'' \]
Summary

Flowchart

$sl(V) \times V \times V^*$ \xrightarrow{H.Ch. descent} $S$ \xrightarrow{Fourier transform and homogeneity theorem} $S'$ \xrightarrow{Fourier transform and integrability theorem} $T'$

$L_{ii} \cap R_A$ \xleftarrow{R_A \cup L_{ij}} $R_A$ \xleftarrow{restriction} $T' - T''$
Summary

Flowchart

\[ sl(V) \times V \times V^* \xrightarrow{\text{H.Ch. descent}} S \xrightarrow{\text{Fourier transform and homogeneity theorem}} S' \xrightarrow{\text{Fourier transform and integrability theorem}} T' \]

\[ \emptyset \xleftarrow{f(R_A \cap L_{ii}) = 0} L_{ii} \cap R_A \xleftarrow{R_A \cup L_{ij}} R_A \xleftarrow{\text{restriction}} T' - T'' \]