EXERCISE 9 IN ALGEBRAIC NUMBER THEORY

(1) (P) The field $\mathbb{Q}_p$ of $p$-adic numbers has no non-trivial automorphisms.

(2) (P) (i) The sequence $1, 1/10, 1/10^2, \cdots$ does not converge in $\mathbb{Q}_p$, for any $p$.

(P) (ii) For every $a \in \mathbb{Z}$, $(a, p) = 1$, the sequence $\{a^p^n\}_{n \in \mathbb{N}}$ converges in $\mathbb{Q}_p$.

(3) ($P^*$) Let $\epsilon \in 1 + p\mathbb{Z}_p$, and let $\alpha = a_0 + a_1p + a_2p^2 \cdots$ be a $p$-adic integer. Let $s_n = a_0 + a_1p + \cdots + a_{n-1}p^{n-1}$. Show that the sequence $\epsilon^{s_n}$ converges to a number $\epsilon^\alpha$ in $1 + p\mathbb{Z}_p$. Show that this turns $1 + p\mathbb{Z}_p$ into a multiplicative $\mathbb{Z}_p$-module.

(4) (P) The fields $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic, unless $p = q$.

(5) ($P^*$) The algebraic closure of $\mathbb{Q}_p$ has infinite degree.