EXERCISE 11 IN ALGEBRAIC NUMBER THEORY

(1) (P) Show that the absolute values of the field \( \mathbb{Q}(\sqrt{5}) \) are given, up to equivalence, as follows.

(i) \(|a + b\sqrt{5}|_1 = |a + b\sqrt{5}|\) and \(|a + b\sqrt{5}|_2 = |a - b\sqrt{5}|\) are the archimedean absolute values.

(ii) If \( p = 2 \) or \( 5 \) or a prime \( \neq 2, 5 \) such that \( (\frac{p}{5}) = -1 \), then there is exactly one extension of \(|\cdot|_p\) to \( \mathbb{Q}(\sqrt{5}) \), namely

\[
|a + b\sqrt{5}|_p = |a^2 - 5b^2|_p^{1/2}.
\]

(iii) If \( p \) is a prime number \( \neq 2, 5 \) such that \( (\frac{p}{5}) = 1 \), then there are two extensions of \(|\cdot|_p\) to \( \mathbb{Q}(\sqrt{5}) \), namely

\[
|a + b\sqrt{5}|_{p_1} = |a + b\gamma|_p, \text{ resp. } |a + b\sqrt{5}|_{p_2} = |a - b\gamma|_p,
\]

where \( \gamma \) is a solution of \( x^2 - 5 = 0 \) in \( \mathbb{Q}_p \).

(2) (i) Show that \( \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}_5 \).

(ii) Find all the solutions of \( x^2 = 14 \mod 625 \).

(iii) How many roots does \( f(x) = x^3 - x - 2 \) have \( \mod 2^{10} \)?

(3) (P*) Show that for a \( p \)-adic field \( K \), every subgroup of finite index in \( K^* \) is both open and closed.

(4) (P*) Show that the maximal unramified extension of \( \mathbb{Q}_p \) is obtained by attaching all the roots of unity of order prime to \( p \). Notice that this infinite extension still has discrete valuation. See Proposition (7.12).

(5) (P*) Let \( \zeta \) be a primitive \( p^m \)-th root of unity. Then one has:

(i) \( \mathbb{Q}_p(\zeta)/\mathbb{Q}_p \) is totally ramified of degree \( \phi(p^m) \), where \( \phi \) is the Euler totient function.

(ii) \( G(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p) \simeq (\mathbb{Z}/p^m\mathbb{Z})^\times \).

(iii) \( \mathbb{Z}_p[\zeta] \) is the valuation ring of \( \mathbb{Q}_p(\zeta) \).

(iv) \( 1 - \zeta \) is a prime element of \( \mathbb{Z}_p[\zeta] \) with norm \( p \).

(v) Let \( \mathfrak{p} \) be the unique maximal ideal. What is the image of \( v_\mathfrak{p}(\mathbb{Z}_p[\zeta]) \)?
(6) (P*) Using questions (4) and (5) above, what can you say about the structure of $G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$, where $\mathbb{Q}_p^{ab}$ denotes the maximal abelian extension of $\mathbb{Q}_p$ in its algebraic closure?