EXERCISE 9 IN ALGEBRAIC NUMBER THEORY

(1) (P) Let $K$ be a number field such that $[K : \mathbb{Q}] = n$. Define a subgroup $M$ of $K$ as

$$M = \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_n,$$

where $\alpha_1, \cdots, \alpha_n$ form a basis of $K/\mathbb{Q}$. Show that the ring of multipliers

$$\mathfrak{o} = \{ \alpha \in K | \alpha M \subseteq M \}$$

is an order in $K$, but in general not the maximal order.

(2) (P) In an order $\mathfrak{o}$ of $K$, the following are equivalent;

(i) $p$ is a non-zero regular prime ideal

(ii) $p$ is invertible

(iii) the set $\{ x \in K | xp \subseteq p \} = \mathfrak{o}$

(3) (P) Let $a$ be an $\mathcal{O}_K$-ideal of $K$. Show that $\mathfrak{o} = \mathbb{Z} + a \mathcal{O}_K$ is an order. Compute the conductor of $\mathfrak{o}$.

(4) ($P^*$) Recall that in a dedekind domain, every invertible ideal is generated by atmost two elements. Is it true for invertible ideals in an order as well?