EXERCISE 6 IN REPRESENTATION THEORY

DMITRY GOUREVITCH

(1) (P) Find the characters of all the irreducible representations of the symmetric group S_4

(2) (P) A representation is called cyclic if it is generated by some vector. Find a necessary and sufficient condition for a representation to be cyclic in terms of multiplicities of isotypic components.

Principal series representations of $\mathrm{SL}(2, \mathbb{F}_q)$

(3) (P) Let $G := \mathrm{SL}(2, \mathbb{F}_q)$ be the group of 2×2 matrices with entries in $\mathbb{F}_q$ and determinant 1. Let $X := \mathbb{F}_q^2 \setminus 0$. Consider the representation of G on $F(X)$, where F is some algebraically closed field of characteristic zero (e.g. $F = \mathbb{C}$). For any character χ of the multiplicative group $\mathbb{F}_q^\times$, let

$$V_\chi := \{f \in F(X) \, s.t. \, f(\lambda x) = \chi(\lambda) f(x)\}$$

Let Δ denote the standard anti-symmetric bilinear form on F^2. Define a linear operator $T : F(X) \to F(X)$ by

$$Tf(x) := \sum_{\{y \, s.t. \, \Delta(x,y)=1\}} f(y).$$

Show that

(a) V_χ is a subrepresentation of $F(X)$ and $F(X) = \bigoplus_\chi V_\chi$

(b) T is an intertwining operator (=morphism of representations) and T maps V_χ into $V_{\chi^{-1}}$.

(c) $\langle F(X), F(X) \rangle = 2q - 2$.

(d) V_χ is irreducible if $\chi \neq \chi^{-1}$ and is a sum of 2 non-isomorphic irreducible components if $\chi = \chi^{-1}$. Find the 2 components. Note that if q is odd there are only two χ with $\chi = \chi^{-1}$, and if q is even there is only one: $\chi = 1$.

(e) $V_\chi \simeq V_\psi$ if and only if $\chi = \psi^{\pm 1}$.

(f) T is invertible

(g) The sum of squares of the dimensions of the irreducible representations we found is $q(q+1)(q+3)/2$, which is approximately $|G|/2$.

The V_χ are called principal series representations and T is called the standard intertwining operator.

Date: January 1, 2018.