EXERCISE 1 IN INTRODUCTION TO REPRESENTATION THEORY

DMITRY GOUREVITCH

A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign (□) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

Groups and G-sets

(1) Let G be a group, i.e. a set with three operations $m : G \times G \to G$, $e : pt \to G$ and $inv : G \to G$ satisfying group axioms. Show that the multiplication map m uniquely determines the map e. Show that the map m uniquely determines the map inv.

Definition 1. *Let X be a G-set.*

(a) We say that X is homogeneous if for any $x, y \in X$ there exists an element $g \in G$ such that $gx = y$.

(b) We say that X is free if stabilizer of every point is trivial (i.e. has one element e).

(c) We say that X is G-torser is it is non-empty, homogeneous and free.

(2) Let $a : G \times X \to X$ be an action of the group G on a set X. It is often convenient to describe the action in terms of the graph map $\Gamma : G \times X \to X \times X$ defined by $\Gamma(g, x) := (x, gx)$. Show that the action is homogeneous if and only if Γ is epimorphic and is free if and only if Γ is monomorphic.

Definition 2. *For a G-set X denote by $A := A_X$ the group of automorphisms of X (i.e. the group of isomorphisms of the G-set X to itself) and by $A := A_X$ the group $\mathrm{Aut}(X)$ of all automorphisms of the G-set X. A homogeneous G-set is called normal if the group A_X acts transitively on the set X.*

(3) (P) Let X be a G-torser. Show that any morphism of G-sets $\nu : X \to X$ is an isomorphism. Show that A_X is isomorphic to G. Pay attention that this isomorphism is **not canonical** - it is only canonically defined up to inner automorphisms.

(4) (P) Show that if X is homogeneous then the action of the group A_X on the set X is free. Describe explicitly the group A_X for the case when $X = G/H$. In particular, show that the homogeneous G-set X is normal if and only if stabilizer subgroups of all points coincide (which means that this subgroup H is a normal subgroup of G).

Definition 3. *Let G be a group and $H < G$ a subgroup. The number $[G : H] := \#(G/H)$ is called the index of the subgroup H (it is a natural number).*

(5) (P)

(a) Show that if we have a tower of subgroups $F < H < G$ then we have a product formula $[G : F] = [G : H][H : F]$.

(b) Let $H < G$ be a subgroup of finite index n. Show that H contains a subgroup N which is normal in G such that $[G : N] \leq n!$.

Date: November 26, 2017.

(6) (P) Fix a group G. Subgroups $H, H_0 \subset$ G are called commensurable if there exists a subgroup D which lies inside H and H_0 and has finite index in both subgroups. Show that this is an equivalence relation. Fix one class C of commensurable subgroups. Show that we can uniquely define a function of relative order $[H : F] \in \mathbb{Q}^+$ on all subgroups in class C which satisfies the product formula and coincides with the index $[H : F]$ if H contains F.

(7) (P) Let p be a prime number and S be a finite p -group, i.e. $\#S = p^r$ for some natural number r.

   (a) Show that if S is non-trivial then its center Z(S) is non-trivial.
   (b) Show that there exists a tower of subgroups $S_0 < S_1 < S_2 < ... < S_r = S$ such that $\#S_i = p^i$. Show that we can choose all the subgroups S_i to be normal in S.
   (c) Let G be a finite group of order n. Suppose we know that the number n has a divisor q which is a power of a prime number. Show that in this case G has a subgroup of order q.

Representation Theory.

(8) Show that the image of any character (i.e. 1-dimensional representation) of a finite group lies inside the unit circle in $\mathbb{C}$.

(9) Let $T : (\pi, V) \to (\tau, W)$ be a morphism of representations. Suppose that it is one-to-one and onto. Show that it is an isomorphism. In other words, show that the inverse linear map T^{-1} commutes with the group action.

(10) Let X be a G-set and $F(X)$ be the space of complex-valued functions on X. Define a representation of G on $F(X)$ by $(\pi(g)f)(x) := f(g^{-1}x)$. Show that it is indeed a representation. Show that the action defined by $(\pi'(g)f)(x) = f(gx)$ does <u>not</u> define a representation for some sets X.

(11) (P) Let $(\pi_1, V_1), (\pi_2, V_2)$ be irreducible representations of a group G. Consider the direct sum (π, V) of these representations. The space V has four G-invariant coordinate subspaces $0, V_1, V_2, V$. Show that the representations π_1 and π_1 are isomorphic if and only if there exists a non-coordinate G-invariant subspace in V (i.e. a subspace distinct from the four subspaces listed above).

(12) (P) Let the group S_3 act on $\mathbb{C}^3$ by permuting the coordinates and let $\mathbb{C}^3_0$ denote the subrepresentation $\{(a, b, c) : a + b + c = 0\}$. Show that it is irreducible.

(13) (P) Show that a finite-dimensional representation π of a group G is completely reducible if and only if for any subrepresentation $\tau \subset \pi$ there exists another subrepresentation $\tau' \subset \pi$ such that $\pi = \tau \oplus \tau'$.

(14) Let V be a representation of a finite group G over $\mathbb{C}$. Let $\langle \cdot, \cdot \rangle$ by an scalar product on V. Define

$$\langle u, v \rangle_G := \sum_{g \in G} \langle gu, gv \rangle$$

Show that $\langle \cdot, \cdot \rangle_G$ is a G-invariant scalar product and that for any subrepresentation $\tau \subset \pi$, the orthogonal complement $\tau^\perp \subset V$ is also a subrepresentation (i.e. is G-invariant).

(15) (P) Define a 2-dimensional representation π of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ over a field of characteristic p by letting the generator of the group to act by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that π is not irreducible, but can not be decomposed as a direct sum of 1-dimensional representations.

(16) ($\Box$)$*$ Show that any irreducible representation of a finite commutative group over $\mathbb{R}$ has dimension 1 or 2.

(17) ($\Box$)$*$ Let (π, V) be an irreducible complex representation of some group G. Suppose we know that the space V has countable dimension. Show that Schur's lemma holds for π, i.e. $\dim \mathrm{Hom}(\pi, \pi) = 1$. (Hint. Prove and use the fact that any operator $A : V \to V$ has a non-empty spectrum, i.e. there exists $\lambda \in \mathbb{C}$ such that the operator $A - \lambda Id$ is not invertible).

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo3.html