EXERCISE 2 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) Let $\pi, \tau \in Rep(G)$ and let $\phi : \pi \to \tau$ be a morphism of representations which is an isomorphism of linear spaces. Show that ϕ is an isomorphism of representations. In other words, show that the linear inverse ϕ^{-1} is also a morphism of representations.
(2) (P) Show that a finite-dimensional representation π of a group G is a direct sum of irreducible representations if and only if for any subrepresentation $\tau \subset \pi$ there exists another subrepresentation $\tau' \subset \pi$ such that $\pi = \tau \oplus \tau'$.
(3) (P) Let G be an infinite group and $H < G$ a subgroup of finite index. Let (π, G, V) be a complex representation of G and $L \subset V$ a G-invariant subspace. Suppose we know that the subspace L has an H-invariant complement. Show that then it has a G-invariant complement.

Definition 1. *If X is a finite G-set we denote by π_X the natural representation of the group G on the space $F(X)$ of functions on X.*

(4) (P) Show that if X, Y are finite G-sets then the intertwining number $\langle \pi_X, \pi_Y \rangle$ equals to the number of G-orbits in the set $X \times Y$ (with respect to the diagonal action $g(x, y) = (gx, gy)$).
(5) Let $\pi \in Rep(G)$ and $\tau \in Rep(H)$. Let π^G denote the space of G-invariant vectors, $\pi^G = \{v \in \pi : \pi(g)v = v \, \forall g \in G\}$. Show that $(\pi \otimes \tau)^{G \times H} = \pi^G \otimes \tau^H$.
(6) Show that every complex matrix A with $A^n = Id$ is diagonalizable.
(7) (*) Let G, H be finite groups. Show that any irrep of $G \times H$ is of the form $\sigma \otimes \rho$, where $\sigma \in Irr(G)$, $\rho \in Irr(H)$.
(8) (*) We showed that $\langle \pi, \tau \rangle = \langle \tau, \pi \rangle$. Is that still true over
   (a) $F = \mathbb{R}$?
   (b) $F = \mathbb{F}_p$?

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo3.html

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