EXERCISE 2 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) Let \( \pi, \tau \in \text{Rep}(G) \) and let \( \phi : \pi \to \tau \) be a morphism of representations which is an isomorphism of linear spaces. Show that \( \phi \) is an isomorphism of representations. In other words, show that the linear inverse \( \phi^{-1} \) is also a morphism of representations.

(2) (P) Show that a finite-dimensional representation \( \pi \) of a group \( G \) is a direct sum of irreducible representations if and only if for any subrepresentation \( \tau \subset \pi \) there exists another subrepresentation \( \tau' \subset \pi \) such that \( \pi = \tau \oplus \tau' \).

(3) (P) Let \( G \) be an infinite group and \( H < G \) a subgroup of finite index. Let \( (\pi, G, V) \) be a complex representation of \( G \) and \( L \subset V \) a \( G \)-invariant subspace. Suppose we know that the subspace \( L \) has an \( H \)-invariant complement. Show that then it has a \( G \)-invariant complement.

**Definition 1.** If \( X \) is a finite \( G \)-set we denote by \( \pi_X \) the natural representation of the group \( G \) on the space \( F(X) \) of functions on \( X \).

(4) (P) Show that if \( X, Y \) are finite \( G \)-sets then the intertwining number \( \langle \pi_X, \pi_Y \rangle \) equals to the number of \( G \)-orbits in the set \( X \times Y \) (with respect to the diagonal action \( g(x, y) = (gx, gy) \)).

(5) Let \( \pi \in \text{Rep}(G) \) and \( \tau \in \text{Rep}(H) \). Let \( \pi^G \) denote the space of \( G \)-invariant vectors, \( \pi^G = \{ v \in \pi : \pi(g)v = v \forall g \in G \} \). Show that \( (\pi \otimes \tau)^{G \times H} = \pi^G \otimes \tau^H \).

(6) Show that every complex matrix \( A \) with \( A^n = Id \) is diagonalizable.

(7) (*) Let \( G, H \) be finite groups. Show that any irrep of \( G \times H \) is of the form \( \sigma \otimes \rho \), where \( \sigma \in \text{Irr}(G) \), \( \rho \in \text{Irr}(H) \).

(8) (*) We showed that \( \langle \pi, \tau \rangle = \langle \tau, \pi \rangle \). Is that still true over

(a) \( F = \mathbb{R} \)?

(b) \( F = \mathbb{F}_p \)?