# EXERCISE 2 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) Let $\pi, \tau \in Rep(G)$ and let $\phi : \pi \to \tau$ be a morphism of representations which is an isomorphism of linear spaces. Show that $\phi$ is an isomorphism of representations. In other words, show that the linear inverse $\phi^{-1}$ is also a morphism of representations.

(2) (P) Show that a finite-dimensional representation $\pi$ of a group $G$ is a direct sum of irreducible representations if and only if for any subrepresentation $\tau \subset \pi$ there exists another subrepresentation $\tau' \subset \pi$ such that $\pi = \tau \oplus \tau'$.

(3) (P) Let $G$ be an infinite group and $H < G$ a subgroup of finite index. Let $(\pi, G, V)$ be a complex representation of G and $L \subset V$ a $G$-invariant subspace. Suppose we know that the subspace $L$ has an $H$-invariant complement. Show that then it has a $G$-invariant complement.

**Definition 1.** *If $X$ is a finite $G$-set we denote by $\pi_X$ the natural representation of the group $G$ on the space $F(X)$ of functions on $X$.*

(4) (P) Show that if $X, Y$ are finite $G$-sets then the intertwining number $\langle \pi_X, \pi_Y \rangle$ equals to the number of $G$-orbits in the set $X \times Y$ (with respect to the diagonal action $g(x, y) = (gx, gy)$).

(5) Let $\pi \in Rep(G)$ and $\tau \in Rep(H)$. Let $\pi^G$ denote the space of $G$-invariant vectors, $\pi^G = \{v \in \pi : \pi(g)v = v \, \forall g \in G\}$. Show that $(\pi \otimes \tau)^{G \times H} = \pi^G \otimes \tau^H$.

(6) Show that every complex matrix $A$ with $A^n = Id$ is diagonalizable.

(7) (*) Let $G, H$ be finite groups. Show that any irrep of $G \times H$ is of the form $\sigma \otimes \rho$, where $\sigma \in Irr(G)$, $\rho \in Irr(H)$.

(8) (*) We showed that $\langle \pi, \tau \rangle = \langle \tau, \pi \rangle$. Is that still true over
   (a) $F = \mathbb{R}$?
   (b) $F = \mathbb{F}_p$?

*URL*: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo2.html

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