

EXERCISE 3 IN INTRODUCTION TO REPRESENTATION THEORY

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- (1) (P) Let (π, V) be an irreducible complex representation of a finite group G . Show that it has an invariant Hermitian form H and that any two such forms are proportional.
- (2) (P) Let G be a finite group and let (π, V) be a finite dimensional representation of G over the field of real numbers R .
 - (a) Show that (π, V) is isomorphic to the dual representation (π^*, V) .
 - (b) Suppose that the group G is commutative and the representation (π, V) is irreducible. Show that $\dim V$ equals 1 or 2 and show that both cases are possible.
 - (c) Give an example of irreducible representations (π, G, V) and (τ, H, L) over the field \mathbb{R} such that the tensor product representation $(\pi \otimes \tau, G \times H, V \otimes L)$ is reducible.
- (3) Define a bilinear form on $\mathcal{A}(G)$ by

$$\langle f, h \rangle := \sum_{g \in G} f(g)h(g^{-1})$$

Show that this form is bilinear, symmetric and non-degenerate.

- (4) The following are equivalent:
 - (a) π is isotypic
 - (b) All irreducible subrepresentations of π are isomorphic
 - (c) If $\pi \simeq \omega \oplus \tau$ with $\langle \omega, \tau \rangle = 0$ then either $\omega = 0$ or $\tau = 0$.
- (5) (P) Classify all irreducible representations of the group G of motions of a cube.
Hint. Use the action of G on faces, edges, vertices and main diagonals of the cube, and on regular tetrahedra inscribed in the cube.
- (6) (P) Show that if X, Y are finite G -sets and χ is a character of G then the intertwining number $\langle \pi_X, \chi \pi_Y \rangle$ equals to the number of G -orbits O in the set $X \times Y$ such that for any point $z \in O$, the restriction $\chi|_{G_z}$ of χ to the stabilizer G_z of z is trivial.
- (7) (P) Let $G = S_n$ denote the group of all permutations of a set I with n elements (symmetric group in n symbols). Consider the natural action of G on the set X of all subsets of I . It has orbits X_0, X_1, \dots, X_n , where X_i consists of all subsets of size i (note that X_0 has one element). Consider the corresponding representations $\pi_i := (\pi_{X_i}, G, F(X_i))$ of the group G and decompose them into irreducible components. Describe these components for representations π_0, π_1, π_2 and compute their dimensions.

URL: <http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo2.html>

Date: November 18, 2015.